Sparsity in Max-plus Algebra And Applications in Multivariate Convex Regression

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June 6-11, 2021

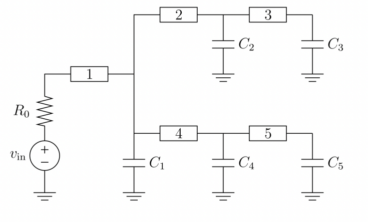
ICASSP2021 TORONTO Canada June 6-11, 2021 Metro Toronto Convention Centre

Overview

Convex regression - why?

Problem: Learn a convex function from data.

Optimization



Machine Learning, Signal Processing & Finance

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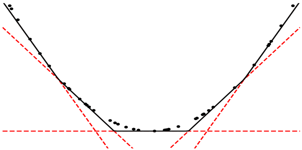

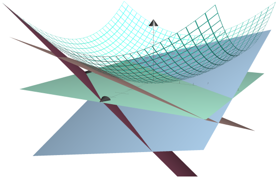

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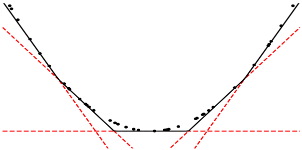

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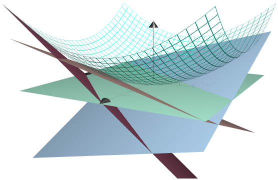

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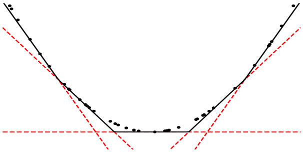

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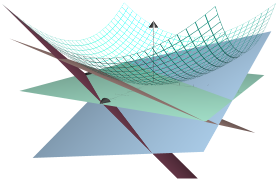

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Step 1. Enforce **sparsity** constraints - Optimization problem:

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Step 2. Account for **noise** - Optimization problem:

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  - Application into the context of convex regression - robust approximations with an approximately *minimum* number of affine regions.

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Theorem (Tsiamis & Maragos 2019)

Computing the sparsest solution of $\mathbf{A} \boxplus \mathbf{b} = \mathbf{y}$ is an NP-complete problem.

Sparse Approximate Solutions (1/3)

Problem formulation

$$\begin{aligned}
&\arg\min_{\mathrm{b}} |\mathrm{supp}(\mathbf{b})| \\
&\quad \text{s.t. } \|\mathbf{y} - \mathbf{A} \boxplus \mathbf{b}\|_p^p \leq \epsilon, \\
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Notes

- We restrict the ℓ_p, $p < \infty$, error to be small
- We add an extra constraint $\mathbf{A} \boxplus \mathbf{b} \leq \mathbf{y}$: Technical reasons & enforce approximation from below!

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Technical details

Tools from Submodular Optimization ⇒ approximation ratio guarantees.

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Proposition

We can find a **locally optimal** *solution of Problem* (2) *by solving Problem* (1).

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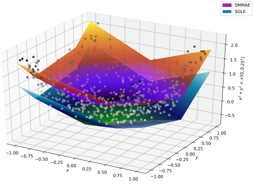
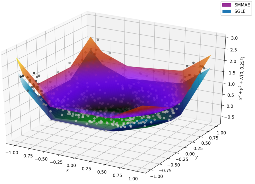
Output: A PWL convex approximation of the data with the approximately *minimum* number of affine regions needed for achieving the desired level of data fidelity.

Data from noisy paraboloid $z = x^2 + y^2 + \mathcal{N}(0,1)$.



(a) Approximation with 5 affine regions.



(b) Approximation with 16 affine regions.

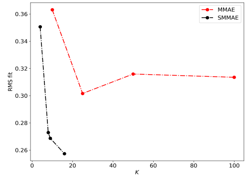


Figure: Comparison of SMMAE method (black line) with [Maragos & Theodosis 2020] (red line).

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Thank you for your attention!

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