Algebraic and PDE Approaches for Multiscale Image Operators with Global Constraints: Reference Semilattice Erosions and Levelings

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Abstract. This paper begins with analyzing the theoretical connections between levelings on lattices and scale-space erosions on reference semilattices. They both represent large classes of self-dual morphological operators that exhibit both local computation and global constraints. Such operators are useful in numerous image analysis and vision tasks ranging from simplification, to geometric feature detection, to segmentation. Previous definitions and constructions of levelings were either discrete or continuous using a PDE. We bridge this gap by introducing generalized levelings based on triphase operators that switch among three phases, one of which is a global constraint. The triphase operators include as special cases reference semilattice erosions. Algebraically, levelings are created as limits of iterated or multiscale triphase operators. The subclass of multiscale geodesic triphase operators obeys a semigroup, which we exploit to find a PDE that generates geodesic levelings. Further, we develop PDEs that can model and generate continuous-scale semilattice erosions, as a special case of the leveling PDE. We discuss theoretical aspects of these PDEs, propose discrete algorithms for their numerical solution which are proved to converge as iterations of triphase operators, and provide insights via image experiments.

1 Introduction

Nonlinear scale-space approaches that are based on morphological erosions and dilations are useful for edge-preserving multiscale smoothing, image enhancement and simplification, geometric feature detection, shape analysis, segmentation, motion analysis, and object recognition. Openings and closings are the basic morphological smoothing filters. The simplest openings/closings, which are compositions of Minkowski erosions and dilations, preserve well vertical image edges but may shift and blur horizontal edges/boundaries. A much more powerful class of filters are the *reconstruction* openings and closings which, starting from a *reference* signal consisting of several parts and a *marker* (initial seed) inside some of these parts, can reconstruct whole objects with exact preservation of their boundaries and edges [13,15]. In this reconstruction process they simplify

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the original image by completely eliminating smaller objects inside which the marker cannot fit. The reference signal plays the role of a *global constraint.* One disadvantage of both the simple as well as the reconstruction openings/closings is that they are not self-dual and hence they treat asymmetrically the image foreground and background. A recent solution to this asymmetry problem came from the development of a more general powerful class of morphological filters, the *levelings* introduced in [8] and further studied in [7,14], which include as special cases the reconstruction openings and closings. The levelings possess many useful algebraic and scale-space properties, as explored in [9], and can be generated by a nonlinear PDE introduced in [6].

A relatively new algebraic approach to self-dual morphology is based not on complete lattices but on inf-semillatices [5]. By using self-dual partial orderings the signal space becomes an inf-semilattice on which self-dual erosion operators can be defined [3,4] that have many interesting properties and applications.

In this paper we develop theoretical connections between levelings on lattices and erosions on semilattices, both from an algebraic and a PDE viewpoint. We begin in Section 2 with a brief background discussion on multiscale operators defined on complete lattices and inf-semilattices. In Section 3 we introduce and analyze algebraically multiscale *triphase* operators (which switch among 3 different states, one state being a global constraint) whose special cases are semilattice erosions and whose limits are levelings. The semigroup of *geodesic* triphase operators is discovered. Afterwards, in Section 4 we model both geodesic levelings and semilattice erosions using PDEs. The main ingredient here is the leveling PDE which we prove it can generate the multiscale geodesic operators and (as a special case) multiscale semilattice self-dual erosions. Section 5 extends the PDE ideas to 2D images signals. In both Sections 4,5 we also propose discrete numerical algorithms for solving the PDEs, prove their convergence using the semilattice operators of previous sections, and provide insights via experiments.

2 Signal Operators on Lattices and Inf-Semilattices

A poset is any set equipped with a partial ordering $\leq$. The supremum ($\bigvee$) and infimum ($\bigwedge$) of any subset of a poset is its lowest upper bound and greatest lower bound, respectively; both are unique if they exist. A poset is called a sup-(inf-) semilattice if the supremum (infimum) of any finite collection of its elements exists. A (sup-) inf-semilattice is called complete if the (supremum) infimum of arbitrary collections of its elements exist. A poset is called a (complete) lattice if it is simultaneously a (complete) sup- and an inf-semilattice. An operator ψ on a complete lattice is called: *increasing* if it preserves the partial ordering $[f \leq g \Longrightarrow \psi(f) \leq \psi(g)]$; idempotent if $\psi^2 = \psi$; antiextensive (extensive) if $\psi(f) \leq f$ ($f \leq \psi(f)$). An operator ε (δ) on a complete inf- (sup-) semilattice is called an *erosion* (*dilation*) if it distributes over the infimum (supremum) of any collection of lattice elements. A *negation* ν is a bijective operator such that both ν and ν^{-1} are either decreasing or increasing and $\nu^2 = \mathsf{id}$, where id is the identity and $\nu \neq \mathsf{id}$. An operator ψ is called *self-dual* if it commutes with a negation ν.

In this paper, the signal space is the collection $\mathbb{V}^{\mathbb{E}}$ of all signals/images defined on $\mathbb{E}$ and assuming values in $\mathbb{V}$, where $\mathbb{E} = \mathbb{R}^d$ or $\mathbb{Z}^d$, $d = 1, 2, ...,$, and $\mathbb{V} \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The value set $\mathbb{V}$ is equipped with some partial ordering that makes it a complete lattice or inf-semilattice. This lattice structure is inherited by the signal space by extending the partial order of $\mathbb{V}$ to signals pointwise. Classical lattice-based mathematical morphology [2] uses as signal space the *complete lattice* $\mathcal{L}(\mathbb{E}, \mathbb{V}) = (\mathbb{V}^{\mathbb{E}}, \vee, \wedge)$ of signals $f : \mathbb{E} \to \mathbb{V}$ with values in $\mathbb{V} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$. In $\mathcal{L}$ the signal ordering is defined by $f \leq g \Leftrightarrow f(x) \leq g(x), \forall x$, and the signal infimum and supremum are defined by $(\bigwedge_i f_i)(x) = \sup_i f_i(x)$ and $\bigvee_i f_i)(x) = \inf_i f_i(x)$. Let B denote henceforth the d-dimensional *unit-radius ball* of $\mathbb{E}$, assuming the Euclidean metric, and let $tB = \{tb : b \in B\}$, $t \geq 0$, be its scaled version. The simplest multiscale dilation/erosion on $\mathcal{L}$ are the *Minkowski flat dilation/erosion* of a signal f by the sets tB:

$$\delta_B^t(f)(x) = (f \oplus tB)(x) = \bigvee_{a \in tB} f(x-a), \quad \varepsilon_B^t(f)(x) = (f \ominus tB)(x) = \bigwedge_{a \in tB} f(x+a) \tag{1}$$

We shall also need the *multiscale conditional dilation and erosion* of a *marker* (‘seed’) signal f within a *reference* (‘mask’) signal r:

$$\delta_{tB}(f|r) := (f \oplus tB) \wedge r, \quad \varepsilon_{tB}(f|r) := (f \ominus tB) \vee r \tag{2}$$

Iterating the conditional dilation (erosion) by a unit-scale B yields the *conditional reconstruction opening (closing)* of r from f.

Another important pair is the geodesic dilation and erosion. First we define them for sets $X \subseteq \mathbb{E}$ (binary images). Let $M \subseteq \mathbb{E}$ a *mask* set and consider its *geodesic metric* $d_M(x, y)$ equal to the length of the *geodesic path* connecting the points x, y inside M. If $B_M(x, t) = \{p \in M : d_M(x, p) \leq t\}$ is the geodesic closed ball with center x and radius $t \geq 0$, then the multiscale geodesic dilation and erosion of X within M are defined by $\delta^t(X|M) := \bigcup_{p \in X} B_M(p, t)$ and $\varepsilon^t(X|M) := [\delta^t(X^c|M^c)]^c$. By using threshold decomposition and synthesis of a signal f from its *threshold sets* $\Theta_h(f) := \{x \in \mathbb{E} : f(x) \geq h\}$ we can synthesize flat geodesic operators for signals by using as generators their set counterparts. The resulting *multiscale geodesic dilation and erosion* of f within a mask signal r are $\delta^t(f|r)(x) := \sup\{h \in \overline{\mathbb{R}} : x \in \delta^t(\Theta_h(f)|\Theta_h(r))\}$ and $\varepsilon^t(f|r)(x) := -\delta^t(-f|-r)$. An equivalent expression is

$$\delta^t(f|r)(x) = r(x) \wedge \bigvee_{d_{M_-}(x,p) \leq t} f(p), \quad \varepsilon^t(f|r)(x) = r(x) \vee \bigwedge_{d_{M_+}(x,p) \leq t} f(p) \tag{3}$$

where $M_- := \{x \in \mathbb{E} : f(x) \leq r(x)\}$ and $M_+ := \{x \in \mathbb{E} : f(x) \geq r(x)\}$. By letting $t \to \infty$ the geodesic dilation (erosion) yields the *geodesic reconstruction opening (closing)* of f within r:

$$\rho^-(f|r) := \bigvee_{t \geq 0} \delta^t(f|r), \quad \rho^+(f|r) := \bigwedge_{t \geq 0} \varepsilon^t(f|r) \tag{4}$$

In [5,3,4] a recent approach for a self-dual morphology was developed based on inf-semilattices. Now, the signal space is the collection of all signals $f : \mathbb{E} \to \mathbb{V}$, where $\mathbb{V} = \mathbb{R}$ or $\mathbb{Z}$. The value set $\mathbb{V}$ becomes a *complete inf-semilattice (cisl)* if equipped with the following partial ordering and infimum:

$$a \preceq_r b \Longleftrightarrow \begin{cases} r \wedge b \le r \wedge a \\ r \vee b \ge r \vee a \end{cases}, \quad \overset{r}{\underset{i}{\curlywedge}} a_i = (r \wedge \bigvee_i a_i) \vee \bigwedge_i a_i$$

for some fixed $r \in \mathbb{V}$. The ordering $\preceq$ coincides with the *activity ordering* in Boolean lattices [10,2].

Given a reference signal $r(x)$, a valid signal cisl ordering is given by

$$f \preceq_r g \iff f(x) \preceq_{r(x)} g(x)\ \forall x \iff |f(x) - r(x) \le |g(x) - r(x)|\ \forall x$$

and the corresponding signal cisl infimum becomes

$$\left(\overset{r}{\underset{i}{\curlywedge}} f_i \right)(x) = [r(x) \wedge \bigvee_i f_i(x)] \vee \bigwedge_i f_i(x) = [r(x) \vee \bigwedge_i f_i(x)] \wedge \bigvee_i f_i(x)$$

Under the above cisl infimum, the signal space becomes a cisl denoted henceforth by $\mathcal{F}_r(\mathbb{E}, \mathbb{V})$, or simply $\mathcal{F}_r$. Among all possible reference cisl's $\mathcal{F}_r$ that result from various choices of the reference signal $r(x)$, the cisl $\mathcal{F}_0$ with $r(x) = 0$ is of primary importance because it is isomorphic to any other $\mathcal{F}_r$. Specifically, the bijection $\xi : \mathcal{F}_0 \to \mathcal{F}_r$, given by $\xi(f) = f + r$, is a cisl isomorphism. Thus, if ψ_0 is an operator on $\mathcal{F}_0$, then its corresponding operator on $\mathcal{F}_r$ is given by

$$\psi_r(f) = \xi \psi_0 \xi^{-1}(f) = r + \psi_0(f - r) \tag{5}$$

If ψ_0 is an erosion on $\mathcal{F}_0$ that is translation-invariant (TI) and self-dual, then ψ_r is also a self-dual TI erosion on $\mathcal{F}_r$. *Note:* the infimum, translation operator and negation operator on $\mathcal{F}_0$ are different from those on $\mathcal{F}_r$. For example, if $\nu_0(f) = -f$ is the negation on $\mathcal{F}_0$, then self-duality of ψ_0 means $\psi_0 \nu_0 = \nu_0 \psi_0$, whereas self-duality on $\mathcal{F}_r$ means $\psi_r \nu_r = \nu_r \psi_r$ where $\nu_r(f) = 2r - f$.

The simplest multiscale TI self-dual erosion on the cisl $\mathcal{F}_r$ is the operator

$$\psi_r^t(f)(x) = r(x) + \left([0 \wedge \bigvee_{a \in tB} (f(x-a) - r(x-a))] \vee \bigwedge_{a \in tB} (f(x-a) - r(x-a)) \right) \tag{6}$$

3 Lattice Levelings and Multiscale Semilattice Erosions

Defining levelings in $\mathcal{L}$ as in [8,7] requires a reference signal r, an input marker signal f, and a *parallel triphase* operator λ_p defined by:

(PT1) $\lambda_p(f, r, \alpha_p, \beta_p) := (r \wedge \beta_p(f)) \vee \alpha_p(f) = (r \vee \alpha_p(f)) \wedge \beta_p(f)$,
(PT2) α_p, β_p are increasing and $\alpha_p(f) \leq f \leq \beta_p(f)$, $\forall f$
where subscript 'p' denotes 'parallel'. In this paper we also define a more general triphase operator, the *serial triphase* operator λ_s, as follows:
(ST1) $\lambda_s(f|r, \alpha_s, \beta_s) := \alpha_s(f|\beta_s(f|r))$,
(ST2) α_s, β_s are increasing, $r \leq \alpha_s(f|r) \leq f \vee r$ and $r \geq \beta_s(f|r) \geq f \wedge r$
where the subscript 's' refers to 'serial' and the operators α_s and β_s have two arguments (f, r), written as $(f|r)$ to emphasize their different roles and provide a slightly different notation from the parallel case. *Any parallel triphase operator becomes a serial one* by setting $\alpha_s(f|r) = \alpha_p(f) \vee r$ and $\beta_s(f|r) = \beta_p(f) \wedge r$. (However, the converse is not always true.) Thus, we henceforth drop the subscripts 's' and 'p' from α, β, λ (the difference will be clear from the context) and focus more on the serial case. The triphase operators depend on four parameters; if some of them are known and fixed, we shall omit them. Thus we may write $\lambda(f|r)$ or simply $\lambda(f)$. A signal f is a called a *parallel (serial) leveling* of r iff it is a fixed point of the parallel (serial) triphase operator, i.e. if $f = \lambda(f|r)$. The original definition in [8,7] corresponds to what we call here parallel leveling.

The definition of the serial triphase operator implies the following.

Proposition 1 *For a serial triphase operator* $\lambda(f|r) = \alpha(f|\beta(f|r))$*:*
(a) $\alpha(r|r) = \beta(r|r) = r$.
(b) $\alpha(f|r) = \alpha(f \vee r|r)$ *and* $\beta(f|r) = \beta(f \wedge r|r)$.
(c) At points where $f \geq r$, $f \geq \lambda(f|r) = \alpha(f|r) \geq r$.
(d) At points where $f \leq r$, $f \leq \lambda(f|r) = \beta(f|r) \leq r$.
(e) α *and* β *commute, i.e.* $\alpha(f|\beta(f|r)) = \beta(f|\alpha(f|r))$.
(f) $r \wedge \lambda = \beta(f|r)$ *and* $r \vee \lambda = \alpha(f|r)$.

Thus, the operator α (β) affects only points where $f \geq r$ ($f \leq r$). Some general properties of triphase operators follow next.

Proposition 2 *(a) Both parallel and serial triphase operators are antiextensive in the cisl* $\mathcal{F}_r$*; i.e.,* $\lambda(f|r) \preceq_r f$.
(b) Let (α_1, β_1) *and* (α_2, β_2) *create two (parallel or serial) triphase operators* λ_1 *and* λ_2*, respectively. If* $\alpha_1 \geq \alpha_2$ *and* $\beta_1 \leq \beta_2$*, then* $\lambda_2(f) \preceq_r \lambda_1(f)$, $\forall f$.
(c) If α *and* β *are dual of each other, then* λ *is self-dual; i.e., if* $\alpha(-f|-r) = -\beta(f|r)$*, then* $\lambda(-f|-r) = -\lambda(f|r)$.

Thus, a leveling of r from the marker f can be obtained by *iterating* any (parallel or serial) triphase operator λ to infinity, or equivalently by taking the cisl infimum $\curlywedge$ of all iterations of λ. Specifically, if $\psi^n(f) := \psi(\cdots\psi(f))$ denotes the n-fold composition of an operator ψ with itself, then

$$\Lambda(f|r) := \lambda^{\infty}(f|r) = \mathop{\curlywedge^{r}}_{n \geq 1} \lambda^n(f) \preceq_r \cdots \preceq_r \lambda^2(f) \preceq_r \lambda(f) \preceq_r f \qquad (7)$$

The map $r \mapsto \Lambda(\cdot|r)$ is called the *leveling operator* and is increasing and idempotent. The signal $g = \Lambda(f|r)$ is obviously a leveling of r from the marker f since $\lambda(g|r) = g$.

If we replace the operators α and β with the multiscale flat erosion and dilation by B of (1) we obtain a *multiscale conditional triphase* operator

$$\lambda_{tB}(f|r)(x) := [r(x) \wedge \delta_{tB}(f)(x)] \vee \varepsilon_{tB}(f)(x) = \underset{a \in tB}{\overset{r}{\curlywedge}} f(x-a) \tag{8}$$

It is called 'conditional' because it can be written as a serial triphase operator, i.e., as a composition of conditional dilation and erosion:

$$\lambda_{tB}(f|r) = \varepsilon_{tB}(f|\delta_{tB}(f|r)) = \delta_{tB}(f|\varepsilon_{tB}(f|r)) \tag{9}$$

Comparing (8) with (6) reveals that λ_{tB} becomes a multiscale TI semilattice erosion on $\mathcal{F}_r$ if r is constant. In particular, if $r=0$, then λ_{tB} becomes a multiscale TI self-dual erosion on $\mathcal{F}_0$. For non-constant r, λ_{tB} is generally neither TI nor an erosion.

By replacing the conditional dilation and erosion in (9) with their geodesic counterparts from (3) we obtain a *multiscale serial geodesic triphase* operator

$$\lambda^t(f|r) = \varepsilon^t(f|\delta^t(f|r)) = \delta^t(f|\varepsilon^t(f|r)) \tag{10}$$

This is the most important triphase operator because it obeys a semigroup. This will allow us later to find its PDE generator.

PROPOSITION 3 *(a) As $t \to \infty$, $\lambda^t(f|r)$ yields the geodesic leveling which is the composition of the geodesic reconstruction opening and closing:*

$$\Lambda(f|r) := \lambda^\infty(f|r) = \rho^-(f|\rho^+(f|r)) = \rho^+(f|\rho^-(f|r)) \tag{11}$$

(b) The multiscale family $\{\lambda^t(\cdot|r) : t \geq 0\}$ forms an additive semigroup:

$$\lambda^t(\cdot|r)\lambda^s(\cdot|r) = \lambda^{t+s}(\cdot|r), \quad \forall t,s \geq 0. \tag{12}$$

(c) For a zero reference ($r=0$), the multiscale geodesic triphase operator becomes identical to its conditional counterpart and the multiscale semilattice erosion:

$$r = 0 \Longrightarrow \psi_0^t(f) = \lambda^t(f|0) = \lambda_{tB}(f|0) \tag{13}$$

(d) For any r, the multiscale semilattice erosion $\psi_r^t = \xi\psi_0^t\xi^{-1}$ obeys a semigroup:

$$\psi_r^t\psi_r^s = \psi_r^{t+s} \quad \forall t,s \geq 0. \tag{14}$$

The above result establishes that, for any positive integer n, the n-th iteration of the unit-scale geodesic triphase operator coincides with its multiscale version at scale $t=n$. The same is true for the multiscale semilattice erosions. It is not generally true, however, for the conditional triphase operator $\lambda_B(f|r)$, which does *not* obey a semigroup. Further, its iterations converge to the *conditional leveling* $\Lambda_B(f|r) = \lambda_B^\infty(f|r)$ which is smaller w.r.t. $\preceq_r$ than the geodesic leveling $\Lambda(f|r) = \lambda^\infty(f|r)$ of (11). Namely, $r \preceq_r \Lambda_B(f|r) \preceq_r \Lambda(f|r)$.

4 PDEs for 1D Levelings and Semilattice Erosions

Consider a 1D reference signal $r(x)$ and a marker signal $f(x)$, both real-valued and defined on $\mathbb{R}$. We start evolving the marker signal by producing the *multiscale geodesic triphase evolutions* $u(x,t) = \lambda^t(f|r)(x)$ of $f(x)$ at scales $t \geq 0$. The initial value is $u_0(x) = u(x,0) = f(x)$. In the limit we obtain the final result $u_\infty(x) = u(x,\infty)$ which will be the leveling $\Lambda(f|r)$. The mapping $u_0 \mapsto u_\infty$ is a *leveling filter*. In [6,9] it was explained that, if $f \leq r$ ($f \geq r$), the leveling $\Lambda(f|r)$ is a reconstruction opening (closing).

In an effort to find a generator PDE for the function u, we shall attempt to analyze the following evolution rule: $\partial u(x,t)/\partial t = \lim_{s\downarrow 0}[u(x,t+s) - u(x,t)]/s$. Since u satisfies the semigroup (12), the evolution rule becomes

$$\frac{\partial u}{\partial t}(x,t) = \lim_{s\downarrow 0} \frac{1}{s}\left[\overset{r}{\underset{|a|\leq s}{\curlywedge}} u(x-a,t) - u(x,t)\right] \tag{15}$$

We shall show later that, at points where the partial derivatives exist this rule becomes the following PDE: $u_t = -\mathrm{sign}(u-r)|u_x|$. However, even if the initial signal f is differentiable, at finite scales $t > 0$, the above switched-erosion evolution may create shocks (i.e., discontinuities in the derivatives). One way to deal with shocks is to replace the standard derivatives with morphological sup/inf derivatives as in [1]. For example, let

$$\mathcal{M}^x u(x,t) := \lim_{s\downarrow 0}[\bigvee_{|a|\leq s} u(x+a,t) - u(x,t)]/s$$

be the sup-derivative of $u(x,t)$ along the x-direction, if the limit exists. If the right $u_x(x+,t)$ and left derivative $u_x(x-,t)$ of u along the x-direction exist, then its sup-derivative also exists and is equal to

$$\mathcal{M}^x u(x,t) = \max[0, u_x(x+,t), -u_x(x-,t)] \tag{16}$$

Obviously, if the left and right derivatives exist and are equal, then the sup-derivative becomes equal to the magnitude $|u_x(x,t)|$ of the standard derivative. The nonlinear derivative $\mathcal{M}$ leads next to a more general PDE that can handle discontinuities in $\partial u/\partial x$.

Theorem 1. [1] *Let $u(x,t) = \lambda^t(f|r)(x)$ be the scale-space function of multiscale geodesic triphase operations with initial condition $u(x,0) = f(x)$. Assume that f is continuous and possesses left and right derivatives at all x. (a) If the partial sup-derivative $\mathcal{M}^x u$ exists at some (x,t), then*

$$\frac{\partial u}{\partial t}(x,t) = \begin{cases} \mathcal{M}^x(u)(x,t), & \text{if } u(x,t) < r(x) \\ -\mathcal{M}^x(-u)(x,t), & \text{if } u(x,t) > r(x) \\ 0, & \text{if } u(x,t) = r(x) \end{cases} \tag{17}$$

[1] Due to space limitations, the proofs of all theorems and propositions will be given in a forthcoming longer paper.

(b) If the partial left and right derivatives $u_x(x\pm,t)$ exist at some (x,t), then

$$\frac{\partial u}{\partial t}(x,t) = \begin{cases} \max[0, u_x(x+,t), -u_x(x-,t)], & \text{if } u(x,t) < r(x) \\ \min[0, u_x(x+,t), -u_x(x-,t)], & \text{if } u(x,t) > r(x) \\ 0, & \text{if } u(x,t) = r(x) \end{cases} \tag{18}$$

(c) If the partial derivative $\partial u/\partial x$ exists at some (x,t), then u satisfies

$$\frac{\partial u}{\partial t}(x,t) = -\text{sign}[u(x,t) - r(x)] \left| \frac{\partial u}{\partial x}(x,t) \right| \tag{19}$$

Thus, assuming that $\partial u/\partial x$ exists and is continuous, the nonlinear PDE (19) can generate the multiscale evolution of the initial signal $u(x,0) = f(x)$ under the action of the triphase operator. However, even if f is differentiable, as the scale t increases, this evolution can create shocks. In such cases, the more general PDE (18) that uses morphological derivatives still holds and can propagate the shocks provided the equation evolves in such a way as to give solutions that are piecewise differentiable with left and right limits at each point.

Consider now on the cisl $\mathcal{F}_0$ the *multiscale TI semilattice erosions* of a 1D signal $f(x)$ by 1D disks $tB = [-t,t]$:

$$v(x,t) = \psi_0^t(f)(x) = [0 \wedge \bigvee_{|a|\leq t} f(x-a)] \vee \bigwedge_{|a|\leq t} f(x-a) \tag{20}$$

This new scale-space function $v(x,t)$ becomes a special case of the corresponding function $u(x,t)$ for multiscale geodesic triphase operations when the reference r is zero. Thus, we can use the leveling PDE (19) with $r(x) = 0$ to generate the evolutions $v(x,t)$:

$$\begin{aligned} \partial v/\partial t &= -\text{sign}(v)|\partial v/\partial x| \\ v(x,0) &= f(x) \end{aligned} \tag{21}$$

If $r(x)$ is not zero, then from the rule (5) that builds operators in $\mathcal{F}_r$ from operators in $\mathcal{F}_0$, we can generate multiscale TI semilattice erosions $\psi_r^t(f) = r + \psi_0^t(f-r)$ of f, defined explicitly in (6), by the following PDE system

$$\psi_r^t(f)(x) = r(x) + v(x,t), \quad \begin{aligned} \partial v/\partial t &= -\text{sign}(v)|v_x| \\ v(x,0) &= f(x) - r(x) \end{aligned} \tag{22}$$

To find a *numerical algorithm* for solving the previous PDEs, let U_i^n be the approximation of $u(x,t)$ on a grid $(i\Delta x, n\Delta t)$). Similarly, define $R_i := r(i\Delta x)$ and $F_i := f(i\Delta x)$. Consider the forward and backward difference operators:

$$D^{+x}U_i^n := (U_{i+1}^n - U_i^n)/\Delta x, \quad D^{-x}U_i^n := (U_i^n - U_{i-1}^n)/\Delta x \tag{23}$$

To produce a shock-capturing and entropy-satisfying numerical method for solving the leveling PDE (19) we approximate the more general PDE (18) by replacing time derivatives with forward differences and left/right spatial derivatives with backward/forward differences. This yields the following algorithm:

$$\begin{aligned} U_i^{n+1} &= U_i^n - \Delta t[\, (P_i^n)^+ \max(0, D^{-x}U_i^n, -D^{+x}U_i^n) \\ &\qquad + (P_i^n)^- \max(0, -D^{-x}U_i^n, D^{+x}U_i^n)] \\ \text{sign}(U_i^{n+1} - R_i) &= \text{sign}(F_i - R_i) \end{aligned} \tag{24}$$

where $P_i^n = \text{sign}(U_i^n - R_i)$, $q^+ = \max(0, q)$, and $q^- = \min(0, q)$. We iterate the above scheme for $n = 1, 2, , ...$ starting from the initial data $U_i^0 = F_i$. For stability, $(\Delta t/\Delta x) \leq 0.5$ is required. The above scheme can be expressed as iteration of a conditional triphase operator Φ acting on the cisl $\mathcal{F}_R(\mathbb{Z}, \mathbb{R})$:

$$
\begin{aligned}
&U_i^{n+1} = \Phi(U_i^n), \quad \Phi(F_i) := [R_i \wedge \beta(F_i)] \vee \alpha(F_i), \\
&\alpha(F_i) = \min[F_i, \theta F_{i-1} + (1-\theta)F_i, \theta F_{i+1} + (1-\theta)F_i], \\
&\beta(F_i) = \max[F_i, \theta F_{i-1} + (1-\theta)F_i, \theta F_{i+1} + (1-\theta)F_i], \quad \theta = \Delta t/\Delta x.
\end{aligned} \tag{25}
$$

By using ideas from methods of solving PDEs corresponding to hyperbolic conservation laws [12], we can easily show that this scheme is conservative and monotone increasing (for $\Delta t/\Delta x < 1$), and hence satisfies the entropy condition.

There are also other possible approximation schemes such as the conservative and monotone scheme proposed in [11] to solve the edge-sharpening PDE $u_t = -\text{sign}(u_{xx})|u_x|$. In order to solve the leveling PDE, we have modified this scheme to enforce the sign consistency condition $\text{sign}(U_i^n - R_i) = \text{sign}(F_i - R_i)$. The final algorithm can be expressed via the iteration of a discrete operator Φ as in (25) but with different α and β:

$$
\begin{aligned}
\alpha(F_i) &= F_i - \theta\sqrt{[\max(F_i - F_{i-1}, 0)]^2 + [\min(F_{i+1} - F_i, 0)]^2}, \\
\beta(F_i) &= F_i + \theta\sqrt{[\min(F_i - F_{i-1}, 0)]^2 + [\max(F_{i+1} - F_i, 0)]^2}
\end{aligned} \tag{26}
$$

This second approximation scheme is more diffusive and requires more computation per iteration than the first scheme (25). Thus, as the main numerical algorithm to solve the leveling PDE, we henceforth adopt the first scheme (25), which is based on discretizing the morphological derivatives. Examples of running this algorithm are shown in Fig. 1. An important question is whether the two above algorithms converge. The answer is affirmative as proved next.

PROPOSITION 4 *If $\Phi(\cdot) = [R \wedge \beta(\cdot)] \vee \alpha(\cdot)$ and (α, β) are either as in (25) or as in (26), the sequence $U^{n+1} = \Phi(U^n)$, $U^0 = F$, converges to a unique limit $U^\infty = \Phi^\infty(F)$ which is a leveling of R from F.*

If $\Delta t = \Delta x$, then Φ of (25) becomes a discrete conditional triphase operator with a unit-scale window $B = \{-1, 0, 1\}$, the PDE numerical algorithm coincides with the iterative discrete algorithm of [8], and the limit of the algorithm is the conditional leveling of R from F.

5 PDEs for 2D Levelings and Semilattice Erosions

A straightforward extension of the leveling PDE from 1D to 2D signals is to replace the 1D dilation PDE with the PDE generating multiscale dilations by a disk. Then the 2D leveling PDE becomes:

$$
\begin{aligned}
u_t(x, y, t) &= -\text{sign}[u(x, y, t) - r(x, y)]||\nabla u(x, y, t)|| \\
u(x, y, 0) &= f(x, y)
\end{aligned} \tag{27}
$$

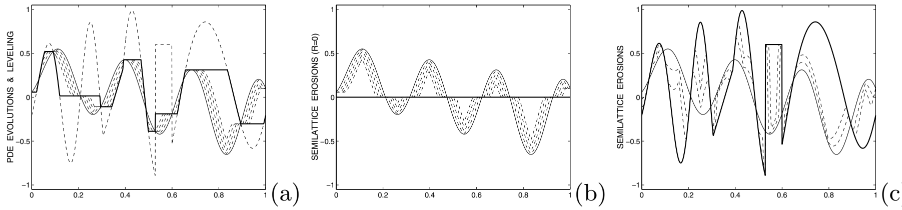

Fig. 1. (a) A reference signal r (dash line), a marker signal m (thin solid line) and its evolutions $u(x,t)$ (thin dash line) generated by the leveling PDE $u_t = -\text{sign}(u-r)|u_x|$, at $t = n25\Delta t$, $n = 1,2,3,4$. (b) Multiscale semilattice erosions $v(x,t)$ of $m(x)$ w.r.t. zero reference, generated by the PDE $v_t = -\text{sign}(v)|v_x|$, $v(x,0) = m(x)$, at $t = n25\Delta t$, $n = 1,2,3,4$. (c) Multiscale semilattice erosions $v(x,t) + r(x)$ of $m(x)$ w.r.t. reference $r(x)$, generated by the PDE $v_t = -\text{sign}(v)|v_x|$, $v(x,0) = m(x) - r(x)$, at $t = n25\Delta t$, $n = 1,2$. ($\Delta x = 0.001$, $\Delta t = 0.0005$.)

Of course, we could select any other PDE modeling erosions by shapes other than the disk, but the disk has the advantage of creating an isotropic growth.

For discretization, let $U_{i,j}^n$ be the approximation of $u(x,y,t)$ on a computational grid $(i\Delta x, j\Delta y, n\Delta t)$ and set the initial condition $U_{ij}^0 = F_{ij} = f(i\Delta x, j\Delta y)$. Then, by replacing the magnitudes of standard derivatives with morphological derivatives and by expressing the latter with left and right derivatives which are approximated with backward and forward differences, we have developed the following entropy-satisfying scheme for solving the 2D leveling PDE (27):

$$\begin{aligned} U_{i,j}^{n+1} &= \Phi(U_{i,j}^n), \quad \Phi(F_{ij}) := [R_{ij} \wedge \beta(F_{ij})] \vee \alpha(F_{ij}), \\ \alpha(F_{ij}) &= F_{ij} - \Delta t \sqrt{\max^2[0, D^{-x}F_{ij}, -D^{+x}F_{ij}] + \max^2[0, D^{-y}F_{ij}, -D^{+y}F_{ij}]} \\ \beta(F_{ij}) &= F_{ij} + \Delta t \sqrt{\max^2[0, -D^{-x}F_{ij}, D^{+x}F_{ij}] + \max^2[0, -D^{-y}F_{ij}, D^{+y}F_{ij}]} \end{aligned} \tag{28}$$

For stability, $(\Delta t/\Delta x + \Delta t/\Delta y) \leq 0.5$ is required. This scheme is theoretically *guaranteed to converge to a leveling*. Examples of running the above 2D algorithm are shown in Fig. 2.

Why use PDEs for levelings and semilattice erosions? In addition to the well-known advantages of the PDE approach (such as more insightful mathematical modeling, more connections with physics, better approximation of Euclidean geometry, and subpixel accuracy), there are also some advantages over the discrete modeling that are specific for the operators examined in this paper. For levelings the desired result is mainly the final limit. The PDE numerical algorithms converge to a leveling Λ_{num}. The discrete (algebraic) algorithm of [8] converges to the conditional leveling Λ_{con}. If Λ is the sampled true (geodesic) leveling, then $r \preceq_r \Lambda_{con} \preceq_r \Lambda_{num} \preceq_r \Lambda$. Hence, the discrete algorithm result has a larger absolute deviation from the true solution than the PDE algorithm. Further, the discrete algorithm uses $\Delta t = \Delta x$ and hence it is unstable (amplifies small errors). In the 2D case we have an additional comparison issue: In

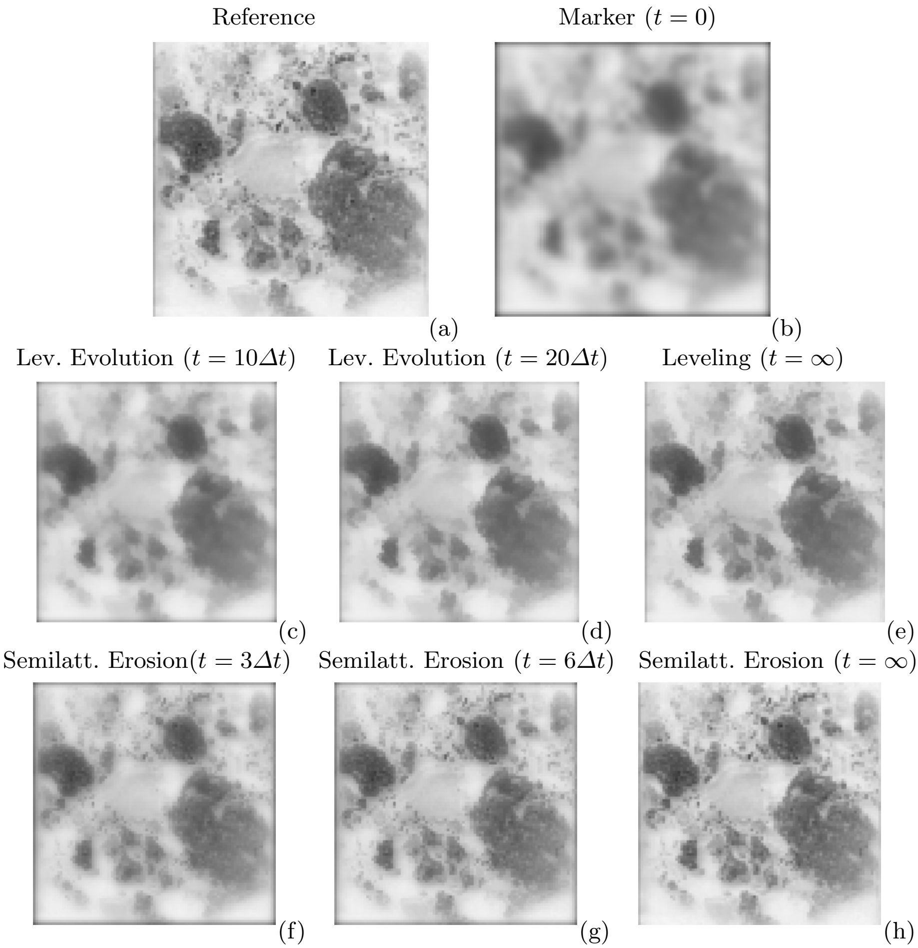

Fig. 2. Multiscale semilattice erosions and levelings of soilsection images generated by PDEs. (a) Reference image $r(x, y)$. (b) Marker image $m(x, y)$ obtained from a 2D convolution of r with a 2D Gaussian of $\sigma = 4$. Images (c),(d),(e) show evolutions $u(x, y, t)$ generated by the leveling PDE $u_t = -\text{sign}(u - r)||\nabla u||$. Images (f),(g),(h) show multiscale semilattice erosions $v(x, y, t) + r(x, y)$ generated by the PDE $v_t = -\text{sign}(v)||\nabla v||$ with $v(x, y, 0) = m(x, y) - r(x, y)$. ($\Delta x = \Delta y = 1$, $\Delta t = 0.25$.)

some applications we may need to stop the marker growth before convergence. In such cases, the isotropy of the partially grown marker offered by the PDE is an advantage.

For multiscale semilattice operators the final limit is not interesting since it coincides with the reference; i.e., $\psi_r^\infty(f) = r$, $\forall f$. What is more interesting in this case are the intermediate results. In this case producing 2D semilattice erosions via the following PDE system yields isotropic results

$$\psi_r^t(f)(x,y) = r(x,y) + v(x,y,t), \qquad \begin{array}{c} \partial v/\partial t = -\text{sign}(v)||\nabla v|| \\ v(x,y,0) = f(x,y) - r(x,y) \end{array} \tag{29}$$

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