8

Morphological Global Reconstruction and Levelings: Lattice and PDE Approaches

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Abstract

This chapter begins with analyzing the theoretical connections between levelings on lattices and scale-space erosions on reference semilattices. They both represent large classes of self-dual morphological reconstruction operators that exhibit both local computation and global constraints. Such operators are useful in numerous image analysis and vision tasks including edge-preserving multiscale smoothing, image simplification, feature and object detection, segmentation, shape, texture and motion analysis. Previous definitions and constructions of levelings were either discrete or continuous using a PDE. We bridge this gap by introducing generalized levelings based on triphase operators that switch among three phases, one of which is a global constraint. The triphase operators include as special cases useful classes of semilattice erosions. Algebraically, levelings are created as limits of iterated or multiscale triphase operators. The subclass of multiscale geodesic triphase operators obeys a semigroup, which we exploit to find PDEs that can generate geodesic levelings and continuous-scale semilattice erosions. We discuss theoretical aspects of these PDEs, propose discrete algorithms for their numerical solution which converge as iterations of triphase operators, and provide insights via image experiments.

8.1 Introduction

Nonlinear scale-space approaches that are based on morphological operators are useful for edge-preserving multiscale smoothing, image simplification, geometric feature detection, segmentation, shape, texture and motion analysis, and object recognition.

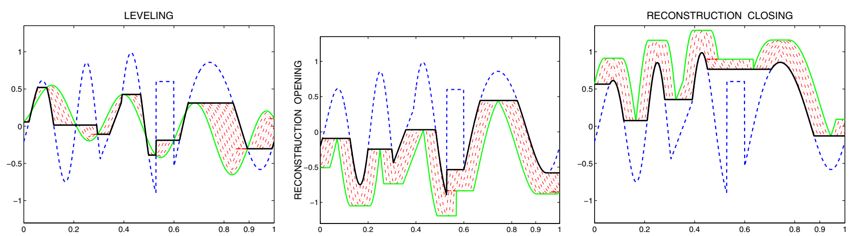
The theory and implementations behind the standard multiscale morphological filters evolved first [347, 467, 333] from a geometric viewpoint that focused on shape-size analysis and an algebraic viewpoint that was based on set theory, level sets and min-max filtering. During the previous decade both the algebraic and geometric aspects of morphology were generalized and improved, by extending its algebra using the theory of complete lattices [468, 233] and by modeling the dynamics and geometry of multiscale morphology using PDEs and curve evolution [9, 70, 459, 336]. The simplest morphological smoothers are translation-invariant (TI) Minkowski openings and closings, i.e., compositions of Minkowski erosions and dilations by compact disk-like sets. These nonlinear smoothers preserve well the edges of the remaining image signal parts but may blur the boundaries of their supports at places where the structuring element cannot fit. The opening increasingly reconstructs some of the image parts lost via erosion, whereas the closing reconstructs by shrinking the parts added via dilation. This reconstruction is *local* and extends only up to the scale of these filters.

A much more powerful class of filters are the *reconstruction* openings and closings which, starting from a *reference* image consisting of several parts and a *marker* (initial seed) inside some of these parts, can reconstruct whole objects with exact preservation of their boundaries and edges [554, 455]. In this *global* reconstruction process they simplify the original image by completely eliminating smaller objects inside which the marker cannot fit. The reference image plays the role of a *global constraint*. One disadvantage of both the simple as well as the reconstruction openings/closings is that they are not self-dual and hence they treat asymmetrically the image foreground vs. background or the bright vs. dark objects. A recent solution to this asymmetry problem came from the development of a more general powerful class of self-dual morphological filters, the *levelings* which include as special cases the reconstruction openings and closings. The levelings were introduced by Meyer [366] for digital spaces and further studied in [348, 469]. They possess many useful algebraic scale-space properties, as explored in [370]. Maragos & Meyer [369] extended them to continuous spaces by generating levelings with the following nonlinear PDE:

$$\begin{aligned} \partial u(x,y,t)/\partial t &= -\text{sign}(u-r)||\nabla u|| \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{8.1}$$

where $u(x,y,t)$ is the scale-space function, $r(x,y)$ is the reference image and $f(x,y)$ is a marker. At scale-space points where $u > r$ (resp., $u < r$), the above PDE generates multiscale erosions (resp., dilations) by disks. The leveling $\Lambda(f|r)$ of r w.r.t. f is produced when $t \to \infty$. In [369, 370] it was explained that, if $f \leq r$ (resp., $f \geq r$), the leveling is a reconstruction opening (resp., closing). Examples are shown in Fig. 8.1.

A relatively new algebraic approach to self-dual morphology was developed by Keshet [269] and Heijmans & Keshet [234, 235] based not on complete lattices but on inf-semillatices. Specifically, by using self-dual partial orderings the image space becomes an inf-semilattice on which self-dual erosion operators can



(a) (b) (c)

Figure 8.1. Evolutions of 1D leveling PDE $u_t = -\text{sign}(u - r)|u_x|$ for 3 different markers $u(x, 0) = f(x)$. Each figure shows the reference signal r (dash line), the marker f (thin solid line), its evolutions $u(x, t)$ (thin dashdot line) at $t = n25\Delta t$, $n = 1, 2, ...,$ and the leveling $u(x, \infty)$ (thick solid line). The 3 markers f were: (a) Arbitrary. (b) An erosion of r minus a constant; hence, the leveling is a reconstruction opening. (c) A dilation of r plus a constant; hence the leveling is a reconstruction closing.

be defined that have many interesting properties and promising applications in nonlinear image analysis.

This chapter continues with an intuitive discussion of multiscale levelings and their interpretation using level sets. Then, after a brief background on lattice operators, we analyze algebraically multiscale *triphase* operators, whose limits are levelings. Special cases of these triphase operators are semilattice erosions. The semigroup of *geodesic* triphase operators is emphasized. Afterwards, we focus on PDEs that can generate both geodesic levelings and TI semilattice self-dual erosions We discuss theoretical aspects of these PDEs, propose algorithms for their numerical solution which converge as iterations of discrete triphase operators, and provide insights via image experiments. The proofs of all propositions and theorems of this chapter can be found in a recent paper by Maragos [335].

8.2 Multiscale Levelings and Level Sets

Consider a reference image r and a leveling Λ. If we can produce various markers f_i, $i = 1, 2, 3, ...,$ that are related to some increasing scale parameter i, let us construct the levelings $g_i = \Lambda(f_i|g_{i-1})$, $i = 1, 2, 3, ...,$ with $g_0 = r$. The signals g_i constitute a hierarchy of *multiscale levelings* possessing the causality property that g_j is a leveling of g_i for $j > i$. One way to construct such multiscale levelings is to use a sequence of multiscale markers obtained from sampling a Gaussian scale-space. As shown in Fig. 8.2, the image edges and boundaries which have been blurred and shifted by the Gaussian scale-space are better preserved across scales by the multiscale levelings.

The main approach in mathematical morphology to extend set operators to image function operators is using level sets and threshold superposition [467, 333,

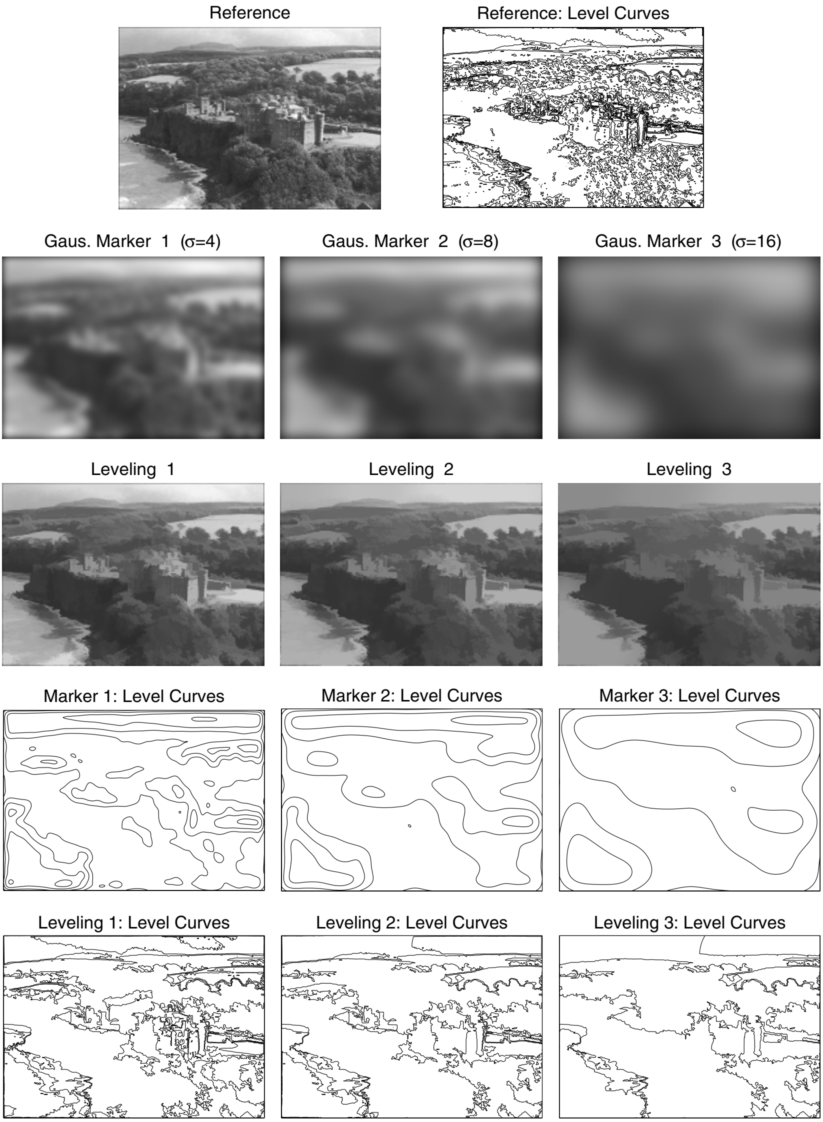


Figure 8.2. Multiscale image levelings $u(x, y, \infty)$ generated by the PDE $u_t = -\text{sign}(u - r)||\nabla u||$. The markers $u(x, y, 0) = f(x, y)$ were obtained by convolving the reference image with 2D Gaussians of standard deviations $\sigma_1 = 4$, $\sigma_2 = 8$, $\sigma_3 = 16$. At each scale σ_i as reference r was used the leveling of the previous scale σ_{i-1}. The last two rows show level curves of the markers and the corresponding levelings (6 level curves for each image).

233]. Specifically, if

$$X_h(f) \triangleq \{x : f(x) \geq h\}, \quad -\infty < h < \infty \tag{8.2}$$

are the upper *level sets* of a real image function f, and we are given an increasing set operator Ψ, then we can construct a so-called *flat operator* $\psi(f) = \sup\{h : x \in \Psi(X_h(f))\}$ via *threshold superposition* of the outputs of the set operator acting on all input level sets. The flat operator ψ can process both binary and graylevel images. The levelings produced by the PDE (8.1) are flat operators and hence they can be constructed by their set counterparts via threshold superposition. Thus,

$$\Lambda(f|r) = \sup\{h : x \in \Lambda(X_h(f)|X_h(r))\} \tag{8.3}$$

The set equivalent of the PDE (8.1) is a curve evolution that propagates the level curves of u with normal speed ± 1 whose sign corresponds to $\mathrm{sign}(r-u)$. Namely, when $u > r$ ($u < r$), the level curves are moved toward (opposite) the direction of the gradient of u. In the limit, the curves will converge to the level curves of the leveling. The level curves of the multiscale levelings are shown in Fig. 8.2 to preserve the global boundaries of image regions much more accurately than the multiscale Gaussian convolutions.

8.3 Multiscale Image Operators on Lattices

A poset is any set equipped with a partial ordering $\leq$. The supremum $(\bigvee)$ and infimum $(\bigwedge)$ of any subset of a poset is its lowest upper bound and greatest lower bound, respectively; both are unique if they exist. A poset is called a (sup-) inf-semillattice if the (supremum) infimum of any finite collection of its elements exists. A (sup-) inf-semilattice is called complete if the (supremum) infimum of arbitrary collections of its elements exist. A poset is called a (complete) lattice if it is simultaneously a (complete) sup- and an inf-semilattice. An operator ψ on a complete lattice is called: *increasing* if it preserves the partial ordering $[f \leq g \Longrightarrow \psi(f) \leq \psi(g)]$; *idempotent* if $\psi^2 = \psi$; *antiextensive (resp., extensive)* if $\psi(f) \leq f$ (resp., $f \leq \psi(f)$). An operator ε (resp., δ) on a complete inf-semilattice (resp., sup-semilattice) is called an *erosion* (resp., *dilation*) if it distributes over the infimum (resp., supremum) of any collection of lattice elements; namely $\delta(\bigvee_i f_i) = \bigvee_i \delta(f_i)$ and $\varepsilon(\bigwedge_i f_i) = \bigwedge_i \varepsilon(f_i)$. A lattice operator is called an *opening* (resp., *closing*) if it is increasing, idempotent, and antiextensive (resp., extensive). A *negation* is a bijective operator $\nu \neq \mathsf{id}$ such that both ν and ν^{-1} are either decreasing or increasing and $\nu^2 = \mathsf{id}$, where id is the identity. An operator is called *self-dual* if it commutes with a negation.

In this chapter, the image space is the collection of signals defined on a continuous or discrete domain $\mathbb{E}$ and assuming values in $\mathbb{V}$, where $\mathbb{E} = \mathbb{R}^m$ or $\mathbb{Z}^m$, $m = 1, 2, ...$, and $\mathbb{V} \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The value set $\mathbb{V}$ is equipped with some partial ordering that makes it a complete lattice or inf-semilattice. This lattice structure is inherited by the image space by extending the partial order of $\mathbb{V}$ to signals pointwise.

8.3.1 Multiscale Operators on Complete Lattices

Classical lattice-based morphology [233, 468] uses as image space the *complete lattice* $\mathcal{L}$ of signals $f : \mathbb{E} \to \mathbb{V}$ with values in $\mathbb{V} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$. In $\mathcal{L}$ the signal ordering is defined by $f \leq g \Leftrightarrow f(x) \leq g(x), \forall x$, and the signal infimum and supremum are defined by $(\bigwedge_i f_i)(x) = \sup_i f_i(x)$ and $(\bigvee_i f_i)(x) = \inf_i f_i(x)$. Assume first $\mathbb{E} = \mathbb{R}^m$. Let $B = \{x : ||x|| \leq 1\}$ denote the unit-radius ball in $\mathbb{R}^m$ w.r.t. the Euclidean metric $|| \cdot ||$ and let $tB = \{tb : b \in B\}$ be its version at scale $t \geq 0$. The simplest multiscale dilation/erosion on $\mathcal{L}$ are the *Minkowski flat dilation/erosion* of an image f by tB:

$$\begin{aligned} \delta_B^t(f)(x) &\triangleq (f \oplus tB)(x) = \textstyle\bigvee_{a \in tB} f(x-a) \\ \varepsilon_B^t(f)(x) &\triangleq (f \ominus tB)(x) = \textstyle\bigwedge_{a \in tB} f(x+a) \end{aligned} \tag{8.4}$$

We shall also need the *multiscale conditional dilation and erosion* of a *marker* ('seed') image f given a *reference* ('mask') image r:

$$\delta_{tB}(f|r) \triangleq (f \oplus tB) \wedge r, \quad \varepsilon_{tB}(f|r) \triangleq (f \ominus tB) \vee r \tag{8.5}$$

Iterating the unit-scale conditional dilation (erosion) yields the *conditional reconstruction opening (closing)* of r from f:

$$\begin{aligned} \rho_B^-(f|r) &\triangleq \delta_B^\infty(f|r) = \textstyle\bigvee_{n \geq 1} \delta_B^n(f|r) \\ \rho_B^+(f|r) &\triangleq \varepsilon_B^\infty(f|r) = \textstyle\bigwedge_{n \geq 1} \varepsilon_B^n(f|r) \end{aligned} \tag{8.6}$$

where, for any operator ψ and any positive integer n, ψ^n denotes the n-fold composition of ψ with itself and $\psi^\infty = \lim_{n \to \infty} \psi^n$ if the limit exists.

Another important pair is the geodesic dilation and erosion. First we define them for sets $X \subseteq \mathbb{E}$. Let $R \subseteq \mathbb{E}$ be a *reference* (mask) set and consider its *geodesic metric* $d_R(x, y)$ equal to the length of the geodesic path connecting the points x and y inside R. If $B_R(x, t) = \{p \in R : d_R(x, p) \leq t\}$ is the geodesic closed ball with center x and radius $t \geq 0$, then the multiscale geodesic set dilation of X given R is defined by $\Delta^t(X|R) \triangleq \bigcup_{p \in X} B_R(p, t)$. By using threshold decomposition and synthesis of an image f from its level sets $X_h(f)$ we can synthesize a flat geodesic dilation for images by using as generator its set counterpart. Then, a possible definition of geodesic erosion is via negation. The resulting *multiscale geodesic dilation and erosion* of f given a reference image r are

$$\begin{aligned} \delta^t(f|r)(x) &\triangleq \sup\{h \leq r(x) : x \in \Delta^t(X_h(f)|X_h(r))\} \\ \varepsilon^t(f|r)(x) &\triangleq -\delta^t(-f|-r) \end{aligned} \tag{8.7}$$

The geodesic dilation and erosion possess a semigroup property:

$$\delta^t \delta^s = \delta^{t+s}, \quad \varepsilon^t \varepsilon^s = \varepsilon^{t+s}, \quad \forall s, t \geq 0 \tag{8.8}$$

whereas their conditional counterparts do not: $\delta_{tB}(\delta_{sB}(f|r)|r) \neq \delta_{(t+s)B}(f|r)$. By letting $t \to \infty$ the geodesic dilation (erosion) yields the *geodesic reconstruc-*

tion opening ρ^- (closing ρ^+) of r from f:

$$\begin{array}{rcl} \rho^-(f|r) & \triangleq & \delta^\infty(f|r) = \bigvee_{t\geq 0} \delta^t(f|r) \\ \rho^+(f|r) & \triangleq & \varepsilon^\infty(f|r) = \bigwedge_{t\geq 0} \varepsilon^t(f|r) \end{array} \tag{8.9}$$

The above limit δ^∞ (ε^∞) can also be reached using iterations δ^n (ε^n) for $n \to \infty$ since, due to the semigroup property, the geodesic dilation (erosion) at integer scales $t = n$ can be obtained via n-fold iteration of the unit-scale operator.

8.3.2 Image Operators on Reference Semilattices

In [269, 234, 235] a recent approach for a self-dual morphology was developed based on inf-semilattices. Now, the image space is the collection of signals $f : \mathbb{E} \to \mathbb{V}$, where $\mathbb{V} = \mathbb{R}$ or $\mathbb{Z}$. The value set $\mathbb{V}$ becomes a *complete inf-semilattice (cisl)* if we select an arbitrary *reference* element $r \in \mathbb{V}$ and use the following partial ordering

$$a \preceq_r b \Longleftrightarrow r \wedge b \leq r \wedge a \;\; \text{and} \;\; r \vee b \geq r \vee a \tag{8.10}$$

which coincides with the *activity ordering* in Boolean lattices [365, 233].

Given a reference image $r(x)$, a valid signal cisl ordering is $f \preceq_r g$ defined as $f(x) \preceq_{r(x)} g(x)\ \forall x$. The corresponding signal cisl infimum becomes

$$\begin{array}{rcl} \left(\curlywedge_i f_i\right)(x) & \triangleq & [r(x) \wedge \bigvee_i f_i(x)] \vee \bigwedge_i f_i(x) \\ & = & \operatorname{med}\left[r(x), \bigvee_i f_i(x), \bigwedge_i f_i(x)\right] \end{array} \tag{8.11}$$

where $\operatorname{med}(\cdot)$ denotes the median. Under the above cisl infimum, the image space becomes a cisl denoted henceforth by $\mathcal{F}_r$. Varying the reference signal r yields cisl's that are all isomorphic to each other. Significant in this chapter is the cisl $\mathcal{F}_0$ with $r(x) = 0$. An isomorphism between $\mathcal{F}_0$ and an arbitrary cisl $\mathcal{F}_r$ is the bijection $\xi(f) = f + r$. Thus, if ψ_0 is an operator on $\mathcal{F}_0$, then its corresponding operator on $\mathcal{F}_r$ is given by $\psi_r(f) = \xi\psi_0\xi^{-1}(f)$. If ψ_0 is an erosion on $\mathcal{F}_0$ that is translation-invariant (TI) and self-dual, then ψ_r is also a self-dual TI erosion on $\mathcal{F}_r$. *Note:* the infimum, translation operator and negation operator on $\mathcal{F}_0$ are different from those on $\mathcal{F}_r$. For example, if $\nu_0(f) = -f$ is the negation on $\mathcal{F}_0$, then self-duality of ψ_0 means $\psi_0\nu_0 = \nu_0\psi_0$, whereas self-duality on $\mathcal{F}_r$ means $\psi_r\nu_r = \nu_r\psi_r$ where $\nu_r(f) = 2r - f$.

The simplest *multiscale TI self-dual erosion* on the cisl $\mathcal{F}_0$ is the operator

$$\psi_0^t(f)(x) = [0 \wedge \bigvee_{a\in tB} f(x-a)] \vee \bigwedge_{a\in tB} f(x-a) \tag{8.12}$$

The corresponding multiscale TI self-dual erosion on the cisl $\mathcal{F}_r$ is

$$\psi_r^t(f) = r + \psi_0^t(f - r) \tag{8.13}$$

8.4 Multiscale Triphase Operators and Levelings

DEFINITION 4 . Given two operators α and β from $\mathcal{L}^2$ to $\mathcal{L}$ that are increasing w.r.t. both arguments and, $\forall f, r$,

$$f \wedge r \leq \beta(f|r) \leq r \leq \alpha(f|r) \leq f \vee r, \tag{8.14}$$

a *triphase* operator λ is defined by

$$\lambda(f|r; \alpha, \beta) \triangleq \alpha(f|\beta(f|r)), = \beta(f|\alpha(f|r)) \tag{8.15}$$

where the operators α and β are assumed to *commute* in the definition (8.15). (Sufficient conditions are given in [335].) The triphase operators have four arguments: two signals f and r and two operators α and β. The signal arguments (f, r) are written as $(f|r)$ to emphasize their asymmetric roles (marker vs. reference) and to connect them with conditional operators that use the same notation. If the operators α and β are known and fixed, we shall omit them and write $\lambda(f|r)$, or simply $\lambda(f)$ if r is assumed. The prototypical example of a triphase operator is when $\alpha(f|r) = \varepsilon_B(f) \vee r$ is a conditional erosion and $\beta(f|r) = \delta_B(f) \wedge r$ is a conditional dilation.

As shown in [335], the action of a triphase operator λ at points x where $f(x) > r(x)$ (resp., $f(x) < r(x)$) is determined only by α (resp., β):

$$\lambda(f|r)(x) = \begin{cases} \beta(f|r)(x), & \text{if} \quad f(x) < r(x) \\ \alpha(f|r)(x), & \text{if} \quad f(x) < r(x) \\ r(x), & \text{if} \quad f(x) = r(x) \end{cases} \tag{8.16}$$

Some general properties of triphase operators follow next.

PROPOSITION 4 ([335]). *(a)* λ *is antiextensive in the cisl* $\mathcal{F}_r$*:* $\lambda(f|r) \preceq_r f$.
(b) λ *is increasing in* $\mathcal{F}_r$*; i.e.,* $f \preceq_r g \Longrightarrow \lambda(f|r) \preceq_r \lambda(g|r)$.
(c) If α *and* β *are dual of each other, then* λ *is self-dual; i.e., if* $\alpha(-f| - r) = -\beta(f|r)$*, then* $\lambda(-f| - r) = -\lambda(f|r)$.

On digital spaces, Meyer [366] and Matheron [348] defined as a leveling of r any signal f such that $\delta_B(f) \wedge r \leq f \leq \varepsilon_B(f) \vee r$. This is equivalent to $f = \lambda(f|r)$, where λ is the conditional triphase formed by the conditional erosion $\varepsilon_B(\cdot|\cdot)$ and dilation $\delta_B(\cdot|\cdot)$ in place of α and β. By generalizing the triphase, we propose the following alternative definition of levelings, valid for discrete or continuous spaces.

DEFINITION 5 . A signal f is a called a λ-induced *leveling* of r iff it is a fixed point of the triphase operator λ, i.e. $f = \lambda(f|r)$.

The following is a necessary and sufficient condition for f to be a λ-induced leveling of r:

$$f = \lambda(f|r) \Longleftrightarrow \beta(f|r) \leq f \leq \alpha(f|r) \tag{8.17}$$

Since any triphase operator λ is antiextensive, a leveling of a reference r from a marker f can possibly be obtained by *iterating* λ to infinity, or equivalently by taking the cisl infimum of all iterations of λ. The limit of these iterations

$$\Lambda(f|r) \triangleq \lambda^{\infty}(f|r) = \curlywedge_{n\geq 1}\lambda^{n}(f) \preceq_r \cdots \preceq_r \lambda(f) \preceq_r f \tag{8.18}$$

exists in the cisl $\mathcal{F}_r$. Henceforth, we shall deal only with triphase operators λ for which $\lambda\lambda^{\infty} = \lambda^{\infty}$; e.g., this happens if α is a lattice erosion and β is a lattice dilation [335]. In such cases, $\Lambda(f|r)$ is a leveling, and the map $r \mapsto \Lambda(f|r)$ is called a *leveling operator*. Note that, Λ is an increasing, antiextensive and idempotent operator, and hence a *semilattice opening*, in the cisl $\mathcal{F}_r$. It is also an increasing and idempotent operator, and hence a *morphological filter*, in the complete lattice $\mathcal{L}$.

If we replace the operators α and β with the multiscale conditional flat erosion and dilation by B of (8.5) we obtain a *multiscale conditional triphase* operator

$$\lambda_{tB}(f|r) \triangleq \varepsilon_{tB}(f|\delta_{tB}(f|r)) = \delta_{tB}(f|\varepsilon_{tB}(f|r)) \tag{8.19}$$

By replacing the conditional dilation and erosion in (8.19) with their geodesic counterparts from (8.7) we obtain a *multiscale geodesic triphase* operator

$$\lambda^{t}(f|r) \triangleq \varepsilon^{t}(f|\delta^{t}(f|r)) = \delta^{t}(f|\varepsilon^{t}(f|r)) \tag{8.20}$$

For both the conditional and the geodesic triphase, its constituent erosion and dilation operators satisfy the basic properties of the operators α and β required for the definitions of triphase operators. Further, they commute.

Comparing (8.19) with (8.13) reveals that $\lambda_{tB}(\cdot|r)$ becomes a multiscale *translation-invariant (TI)* semilattice erosion on $\mathcal{F}_r$ if r is constant. In particular, if $r = 0$, then λ_{tB} becomes a multiscale TI self-dual erosion on $\mathcal{F}_0$. For nonconstant r, λ_{tB} is generally neither TI nor an erosion. In constrast, the geodesic triphase is a semilattice erosion, although *not* TI.

PROPOSITION 5 ([335]). *The geodesic triphase operator* $\lambda^{t}(f|r) = \varepsilon^{t}(f|\delta^{t}(f|r)$ *is a semilattice erosion in the cisl* $\mathcal{F}_r$*; i.e.,* $\lambda^{t}(\curlywedge_i f_i|r) = \curlywedge_i \lambda(f_i|r)$.

The geodesic triphase is the most important triphase operator because it obeys a semigroup. This will allow us later to find its PDE generator.

PROPOSITION 6 ([335]). *(a) As* $t \to \infty$, $\lambda^{t}(f|r)$ *yields the geodesic leveling which is the composition of the geodesic reconstruction opening and closing:*

$$\Lambda(f|r) \triangleq \lambda^{\infty}(f|r) = \rho^{-}(f|\rho^{+}(f|r)) = \rho^{+}(f|\rho^{-}(f|r)) \tag{8.21}$$

(b) The multiscale family $\{\lambda^{t}(\cdot|r) : t \geq 0\}$ *forms an additive semigroup:*

$$\lambda^{t}(\lambda^{s}(\cdot|r)|r) = \lambda^{t+s}(\cdot|r), \quad \forall t, s \geq 0. \tag{8.22}$$

(c) For a zero reference ($r = 0$*), the multiscale geodesic triphase operator becomes identical to its conditional counterpart and the multiscale TI semilattice erosion:* $\psi_0^t(f) = \lambda^{t}(f|0) = \lambda_{tB}(f|0)$.

(d) For any r, the multiscale TI semilattice erosion $\psi_r^t(f) = r + \psi_0^t(f - r)$ *obeys a semigroup:* $\psi_r^t \psi_r^s = \psi_r^{t+s}$.

From the semigroup property (8.22), the n-fold iteration of the unit-scale geodesic triphase operator concides with its multiscale version at integer scale $t = n$. The same is true for the multiscale TI semilattice erosions. It is not generally true, however, for the conditional triphase operator $\lambda_B(f|r)$, which does *not* obey a semigroup. Further, its iterations converge to the *conditional leveling* $\Lambda_B(f|r) = \lambda_B^\infty(f|r)$ which is smaller w.r.t. $\preceq_r$ than the geodesic leveling $\Lambda(f|r) = \lambda^\infty(f|r)$.

8.5 Partial Differential Equations

8.5.1 *PDEs for 1D Levelings and Semilattice Erosions*

Consider a 1D reference image $r(x)$ and a marker image $f(x)$ on $\mathbb{R}$, both real and continuous. We start evolving the marker image by producing the *multiscale geodesic triphase evolutions*

$$u(x,t) = \lambda^t(f|r)(x) = \delta^t(f|\varepsilon^t(f|r))(x) \tag{8.23}$$

The initial value is $u(x, 0) = f(x)$. In the limit we obtain the final result $u(x, \infty)$ which will be the leveling $\Lambda(f|r)$. Attempting to find a generator PDE for the function u, we shall analyze the following evolution rule: $\partial u(x,t)/\partial t = \lim_{s\downarrow 0}[u(x, t + s) - u(x, t)]/s$. By using the semigroup (8.22) that u satisfies and the pointwise representation (8.16) of triphase operators, the evolution rule becomes

$$\frac{\partial u}{\partial t} = \begin{cases} \lim_{s\downarrow 0}[\delta^s(u(x,t)|r)(x) - u(x,t)]/s, & \text{if} \quad u(x,t) < r(x) \\ \lim_{s\downarrow 0}[\varepsilon^s(u(x,t)|r)(x) - u(x,t)]/s, & \text{if} \quad u(x,t) > r(x) \\ 0, & \text{if} \quad u(x,t) = r(x) \end{cases} \tag{8.24}$$

We shall show later that, at points where the partial derivatives exist, this rule becomes the following PDE: $u_t = -\text{sign}(u - r)|u_x|$. Starting from a continuous marker $f(x)$, the evolutions $u(x, t)$ remain continuous for all x, t. However, even if the initial image f is differentiable, at finite scales $t > 0$, the above triphase evolution may create shocks (i.e., discontinuities in the derivatives). One way to propagate these shocks (as done in solving evolution PDEs of the Hamilton-Jacobi type with level-set methods [401]) is to use conservative monotone difference schemes that pick the correct weak solution satisfying the entropy condition. An alternative way we propose to deal with shocks is to replace the standard derivatives with morphological sup/inf derivatives. For example, let

$$\mathcal{M}_x u(x,t) \triangleq \lim_{s\downarrow 0}[\bigvee_{|a|\leq s} u(x+a,t) - u(x,t)]/s$$

be the *sup-derivative* of $u(x, t)$ along the x-direction, if the limit exists. If the one-sided right derivative $\partial_{+x}u(x, t)$ and left derivative $\partial_{-x}u(x, t)$ of u along the

x-direction exist, then its sup-derivative also exists and is equal to $\mathcal{M}_x u(x,t) = \max[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)]$ Obviously, if the left and right derivatives are equal, then the $\mathcal{M}$ becomes equal to the magnitude $|u_x(x,t)|$ of the standard derivative. The nonlinear derivative $\mathcal{M}$ leads next to a more general PDE that can handle discontinuities in $\partial u/\partial x$.

THEOREM 3 ([335]). *Let $u(x,t) = \lambda^t(f|r)(x)$ be the scale-space function of multiscale geodesic triphase operations with initial condition $u(x,0) = f(x)$. Assume that r is continuous and f is continuous with left and right derivatives at all x. (a) If the partial left and right derivatives $\partial_{\pm x}u$ exist at some (x,t), then*

$$\frac{\partial u}{\partial t}(x,t) = \begin{cases} \max[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)], & \text{if} \quad u(x,t) < r(x) \\ \min[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)], & \text{if} \quad u(x,t) > r(x) \\ 0, & \text{if} \quad u(x,t) = r(x) \end{cases} \tag{8.25}$$

(b) If the two-sided partial derivative $\partial u/\partial x$ exists at some (x,t), then u satisfies

$$\frac{\partial u}{\partial t}(x,t) = -\text{sign}[u(x,t) - r(x)] \left| \frac{\partial u}{\partial x}(x,t) \right| \tag{8.26}$$

Thus, assuming that $\partial u/\partial x$ exists and is continuous, the nonlinear PDE (8.26) can generate the multiscale evolution of the initial image $u(x,0) = f(x)$ under the action of the geodesic triphase operator. However, even if f is differentiable, as the scale t increases, this evolution can create shocks. In such cases, the more general PDE (8.25) that uses morphological derivatives still holds and can propagate the shocks provided the equation evolves in such a way as to give solutions that are piecewise differentiable with left and right derivatives at each point.

Consider now on the cisl $\mathcal{F}_0$ the *multiscale TI semilattice erosions* $v(x,t) = \psi_0^t(f)(x)$ of a real 1D image $f(x)$ by 1D line segments $tB = [-t,t]$, defined in (8.12). Since $v(x,t)$ is the special case of the corresponding function $u(x,t)$ for multiscale geodesic triphase operations when $r = 0$, we can use the leveling PDE (8.26) to generate the evolutions $v(x,t)$:

$$\partial v/\partial t = -\text{sign}(v)|\partial v/\partial x|, \quad v(x,0) = f(x) \tag{8.27}$$

If $r(x)$ is not zero, we can generate multiscale TI semilattice erosions $\psi_r^t(f) = r + \psi_0^t(f - r)$ of f by the following PDE system

$$\begin{aligned} \partial v/\partial t &= -\text{sign}(v)|v_x|, \quad v(x,0) = f(x) - r(x) \\ \psi_r^t(f)(x) &= r(x) + v(x,t) \end{aligned} \tag{8.28}$$

If $f - r$ has both negative and positive values and is non-constant, then as $t \to \infty$ we obtain the reference r; i.e, $\psi_r^\infty(f) = r$ because $\psi_0^\infty(f - r) = 0$.

To find a numerical algorithm for solving the previous PDEs, let U_i^n be the approximation of $u(x,t)$ on a grid $(i\Delta x, n\Delta t))$. Similarly, define $R_i \triangleq r(i\Delta x)$ and $F_i \triangleq f(i\Delta x)$. Consider the forward and backward difference operators:

$$D_{+x}U_i^n \triangleq (U_{i+1}^n - U_i^n)/\Delta x, \quad D_{-x}U_i^n \triangleq (U_i^n - U_{i-1}^n)/\Delta x \tag{8.29}$$

To produce a shock-capturing and entropy-satisfying numerical method for solving the leveling PDE (8.26) we approximate the more general PDE (8.25) by replacing time derivatives with forward differences and left/right spatial derivatives with backward/forward differences. This yields the following algorithm:

$$\begin{aligned} U_i^{n+1} \quad = \quad & U_i^n - \Delta t[\ (P_i^n)^+ \max(0, D_{-x}U_i^n, -D_{+x}U_i^n) \\ & \qquad +(P_i^n)^- \max(0, -D_{-x}U_i^n, D_{+x}U_i^n)] \end{aligned} \tag{8.30}$$

where $P_i^n = \mathrm{sign}(U_i^n - R_i)$, $q^+ = \max(0,q)$, and $q^- = \min(0,q)$. Further, to avoid spurious numerical oscillations around zerocrossings of $f - r$, at each iteration we enforce the sign consistency

$$\mathrm{sign}(U_i^n - R_i) = \mathrm{sign}(F_i - R_i), \quad \forall n, i \tag{8.31}$$

We iterate the above scheme for $n = 1, 2, ...$, starting from the initial data $U_i^0 = F_i$. For *stability*, $(\Delta t/\Delta x) \leq 0.5$ is required.

The above scheme can be expressed as *iteration of a discrete triphase operator* Φ acting on the cisl $\mathcal{F}_R$ of 1D sampled real-valued signals with reference R:

$$U_i^{n+1} = \Phi(U_i^n), \quad \Phi(F_i) \triangleq \alpha(F_i) \vee [R_i \wedge \beta(F_i)], \tag{8.32}$$

where the operators α, β are given by, for $\theta = \Delta t/\Delta x$,

$$\begin{aligned} \alpha(F_i) &= \min[F_i, \theta F_{i-1} + (1-\theta)F_i, \theta F_{i+1} + (1-\theta)F_i], \\ \beta(F_i) &= \max[F_i, \theta F_{i-1} + (1-\theta)F_i, \theta F_{i+1} + (1-\theta)F_i]. \end{aligned} \tag{8.33}$$

By using ideas from methods of solving PDEs corresponding to hyperbolic conservation laws [401], we can easily show that this scheme[1] is conservative and monotone increasing for $\theta = \Delta t/\Delta x \leq 1$. Hence, it satisfies the entropy condition. Examples of running this algorithm are shown in Fig. 8.1. An important question is whether the above algorithm converges. The answer is given affirmative next.

PROPOSITION 7 ([335]). *If* $\Phi(\cdot) = \alpha(\cdot) \vee [R \wedge \beta(\cdot)]$ *and* (α, β) *are as in (8.33), then* Φ *is a parallel triphase operator and the sequence* $U^{n+1} = \Phi(U^n)$, $U^0 = F$, *converges to a unique limit* $U^\infty = \Phi^\infty(F)$. *For digital images* F, R *assuming a finite number of gray levels, the limit* $\Phi^\infty(F)$ *is a conditional leveling of* R *from* F.

If $\Delta t = \Delta x$, then the α and β operators (8.33) of the discrete triphase operator Φ in (8.32) become erosion and dilation, respectively, by a unit-scale window $B = \{-1, 0, 1\}$. Further, the corresponding PDE numerical algorithm coincides with the iterative discrete algorithm of [366] for constructing levelings.

[1] There are also other possible approximation schemes, e.g. the scheme proposed in [400] to solve the edge-sharpening PDE $u_t = -\mathrm{sign}(u_{xx})|u_x|$, which is however more diffusive and requires more computation than the above scheme; see[335].

8.5.2 PDEs for 2D Levelings and Semilattice Erosions

A straighforward extension of the leveling PDE from 1D to 2D images results by replacing the term $-|u_x|$ creating 1D multiscale erosions with the term $-||\nabla u||$ generating multiscale erosions by disks. Then the 2D leveling PDE becomes:

$$\begin{array}{lcl} \partial u(x,y,t)/\partial t & = & -\mathrm{sign}[u(x,y,t)-r(x,y)]||\nabla u(x,y,t)|| \\ u(x,y,0) & = & f(x,y) \end{array} \tag{8.34}$$

As in the 1D case, $u(x,y,t) = \lambda^t(f|r)(x,y)$ is a scale-space function holding the 2D multiscale geodesic triphase evolutions of the marker image $f(x,y)$ within the reference image $r(x,y)$. Of course, we could select any other PDE modeling the intermediate growth kernel by shapes other than the disk, but the disk has the advantage of creating an isotropic growth.

For discretization, let $U_{i,j}^n$ be the approximation of $u(x,y,t)$ on a computational grid $(i\Delta x, j\Delta y, n\Delta t)$ and set the initial condition $U_{ij}^0 = F_{ij} = f(i\Delta x, j\Delta y)$. Then, by replacing the magnitudes of standard derivatives with morphological derivatives and by expressing the latter with left and right derivatives which are approximated with backward and forward differences, we arrive at the following entropy-satisfying scheme for solving the 2D leveling PDE (8.34):

$$\begin{aligned} U_{i,j}^{n+1} &= \Phi(U_{i,j}^n), \quad \Phi(F_{ij}) \triangleq [R_{ij} \wedge \beta(F_{ij})] \vee \alpha(F_{ij}), \\ \alpha(F_{ij}) &= F_{ij} \\ &\quad - \Delta t \sqrt{\overset{2}{\max}[0, D_{-x}F_{ij}, -D_{+x}F_{ij}] + \overset{2}{\max}[0, D_{-y}F_{ij}, -D_{+y}F_{ij}]} \\ \beta(F_{ij}) &= F_{ij} \\ &\quad + \Delta t \sqrt{\overset{2}{\max}[0, -D_{-x}F_{ij}, D_{+x}F_{ij}] + \overset{2}{\max}[0, -D_{-y}F_{ij}, D_{+y}F_{ij}]} \end{aligned} \tag{8.35}$$

For stability, $(\Delta t/\Delta x + \Delta t/\Delta y) \leq 0.5$ is required. As in the 1D case, this scheme converges to a discrete conditional leveling. Examples of running the above 2D algorithm are shown in Fig. 8.3. In all image experiments based on PDEs we used $\Delta x = \Delta y = 1$, $\Delta t = 0.25$ as space-time steps.

As a by-product of the 2D leveling PDE, the multiscale TI semilattice erosions (8.13) of a marker image f by disks B w.r.t. a reference image r can be generated as follows:

$$\begin{array}{lcl} \partial v/\partial t & = & -\mathrm{sign}(v)||\nabla v||, \quad v(x,y,0) = f(x,y) - r(x,y) \\ \psi_r^t(f)(x,y) & = & r(x,y) + v(x,y,t) \end{array} \tag{8.36}$$

8.6 Discussion

We conclude by providing some insights on the behavior of levelings and multiscale semilattice erosions via several image experiments. Then, we also

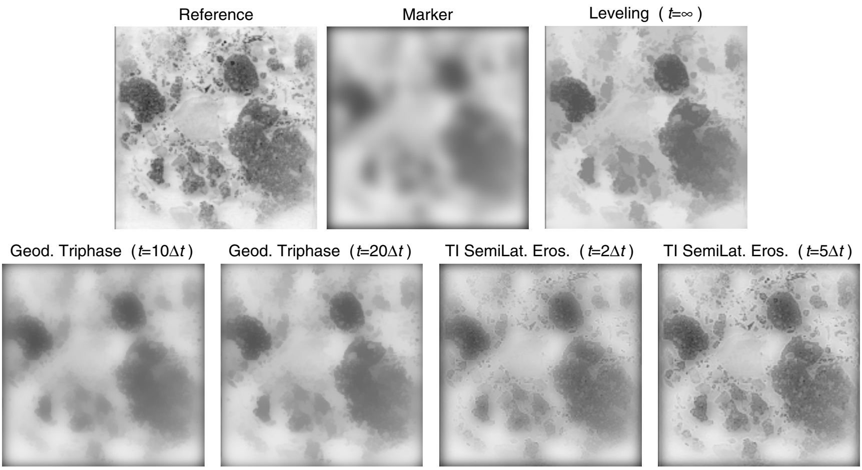


Figure 8.3. Multiscale triphase evolutions and leveling of a soilsection image r generated by PDEs. The marker image f was obtained from a convolution of the reference r with a 2D Gaussian of $\sigma = 8$. Second row: left two images show geodesic triphase evolutions generated by the leveling PDE (8.34); right two images show multiscale TI semilattice erosions generated by the PDE (8.36).

comment on the advantages of PDE-based algorithms for generating these lattice scale-spaces.

As shown in Figs. 8.3 and 8.4, the leveling limit is strongly dominated by the structure of the reference image. Although the selection of markers suitable for producing levelings with various designable properties is still an open issue, it appears that a smooth version of the reference works well as a marker for applications of image simplification and segmentation. This choise also works for image denoising where the marker may be a linear or nonlinear smoothing of the noisy reference. In a different scenario shown in Fig. 8.4, we experimented with a binary edge map as reference whereas the marker was a smooth version of the same original image. Here the intermediate triphase evolutions (geodesic semilattice erosions) toward the leveling seemed useful for adding image region details back to the edge map. Finally, as discussed in [235, 335], the intermediate multiscale TI semilattice erosions seem potentially applicable to mixing or morphing the marker image into the reference, even if the two images are completely unrelated. On comparing the speed of convergence, we have experimentally found that the geodesic triphase evolutions toward levelings converge to the limit more slowly than the multiscale TI semilattice erosions.

The basic *algebraic discrete* algorithm that Meyer [366] developed to construct levelings of digital images is the iteration of the conditional triphase operator $\lambda(F_i) = \varepsilon_B(F_i) \vee [R_i \wedge \delta_B(F_i)]$, where δ_B and ε_B are flat dilation and erosion by a discrete unit-scale disk-like set B. Now we know that this converges to a discrete conditional leveling $\Lambda_{algdiscr}$. The new PDE-based numerical algorithm (8.32),(8.33) converges to another discrete conditional leveling Λ_{pdenum}. If Λ_{true}

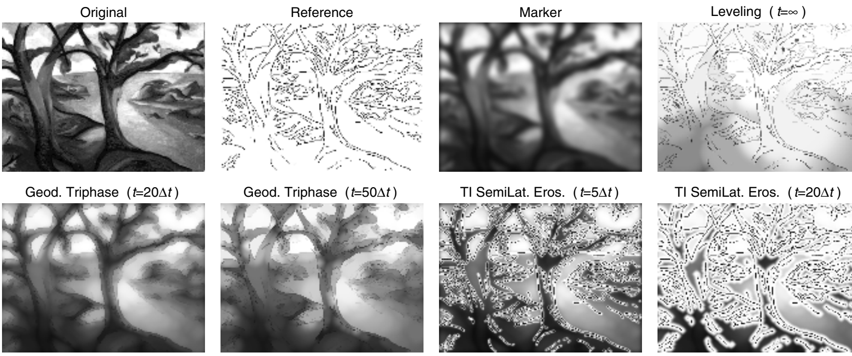


Figure 8.4. First row: original image g, reference image r resulting from applying the Canny edge detector to g, marker image f which is a Gaussian convolution of g, and the leveling $\Lambda(f|r)$. Second row: first two images show multiscale geodesic triphase evolutions (converging to the leveling); last two images show multiscale TI semilattice erosions. (All evolutions were generated by PDEs.)

is the sampled true (geodesic) leveling, then $r \preceq_r \Lambda_{algdiscr} \preceq_r \Lambda_{pdenum} \preceq_r \Lambda_{true}$. Hence, the algebraic discrete algorithm yields a result that has a larger absolute deviation from the true solution than the PDE numerical algorithm. Further, the algebraic discrete algorithm is a special case of the PDE algorithm using the value $\theta = \Delta t/\Delta x = 1$, which makes it *unstable* (amplifies small errors). Thus, in addition to its well-known advantages (such as better geometry and accuracy, physics, and insightful modeling), the PDE approach also has some advantages over the discrete modeling that are specific for the operators examined in this chapter. In the 2D case we have an additional comparison issue: Although for the triphase evolutions toward levelings the desired result in segmentation applications is mainly the final limit, there may be other applications, for instance such as mixing/morphing images, where we need to stop the marker growth before convergence. In such cases as evolutions of 2D multiscale (geodesic or TI) semilattice erosions, the isotropy of the partially grown marker offered by the PDE approach is an advantage over the discrete algebraic approach.