Computer Vision, Speech Communication & Signal Processing Group
National Technical University of Athens, Greece (NTUA)
http://cvsp.cs.ntua.gr

Morphological and Variational Methods in Image Analysis and Vision

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Seminar, Dept. Computer & Information Science, Univ. Pennsylvania, Philadelphia, 05-Oct-2012

CVSP Group Research Areas

1. Image Analysis and Computer Vision
2. Audio, Speech and Language Processing
3. Multimodal Signal/Event Processing and Cognitive Systems
4. Multimodal Human-Robot Communication

Collaborators for work in this talk

- *U+V decomposition, PDE Watershed+Energy Segmentation*:

  Georgios Evangelopoulos (Univ. Houston), Anastasia Sofou

- *Modulation features, Weighted Curve Evolution, Texture Segmentation*:

  Iasonas Kokkinos (Ecole Central Paris), G. Evangelopoulos

- *GACs on Graphs*:

  Kimon Drakopoulos (MIT)

- *Patch-based PDEs & Tensor Diffusions*:

  Anastassios Roussos (Queen Mary Univ. London)

Outline

- Overview of Morphological Operators (Euclidean & Lattice)
- PDEs and Variational Formulations
- U+V: Leveling Cartoons, Texture Energy
- U+V driven PDE-based (W+E) Segmentation
- Unsupervised Texture Segmentation sing AM-FM models and Weighted Curve Evolution
- Patch-based PDEs, Tensor Image Diffusions

Overview of Morphological (Euclidean and Lattice) Concepts

Emphasis on Connected Operators

- **Classic operations**
  - **~1900: Cantor - Minkowski** (Volume Measures): **Minkowski set addition**
  - **1957: Hadwiger** (Integral Geometry): **Minkowski set subtraction**
- **Euclidean Morphology (1960s, 1970s, 1980s →)**
  - **1960s: Binary Morphology, Cellular Logic (Boolean filters,Thresh. Convol.)**
  - **1970s: Gray Morphology via Level Sets, Stochastic Geometry**
  - **1970s: Fuzzy Logic Image Processing: Max-Min filtering**
  - **+ 1980s: Morf/Rank/Stack Filtering, Foundations of Sup/Inf Convolutions Weighted MM),**
  - ***+ Denoising, Feature Extraction, Shape Analysis, Watershed Segmentation***
  - ***+ Morph. Representation Theory**: Every TI Increasing Set (Function) Operator is a Union (Sup) of erosions or an Intersection (Inf) of Dilations.*

    Minimal Representation when using a Basis.
- **Lattice Morphology (1990s →)**
  - **Adjunctions , Fuzzy set MM, + Connected Operators**
  - **Invariance w.r.t. Groups of Generalized Translations.**
- **PDEs for Image Processing and Vision (1990s →)**
  - **+ Differential Morphology: Nonl ScaleSpaces, PDEs, Variational (2000s)**
  - **+ Slope Transforms: Convex analysis, Distance transforms**
  - **+ Curve Evolution: Level Sets, Hamilton-Jacobi PDEs**

    Active Contours: Balloon force, Curvature motion
- **Minimax Algebra: (Intro.Theory, 1980s), + Weighted Lattices (2000s→)**
- **Graph Morphology: (Opers., 1990s), + PdEs, Segm. & (2000s →)**

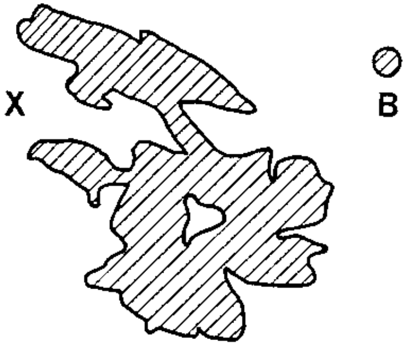
EUCLIDEAN MORPHOLOGICAL SET OPERATORS

Translation: $B_{+z} = \{b+z : b \in B\}$ **Symmetric:** $B^s = \{-b : b \in B\}$

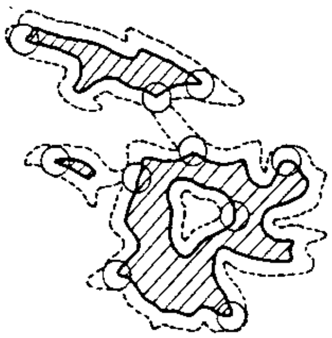
Dilation (Minkowski addition): $X \oplus B = \{z : (B^s)_{+z} \cap X \neq \varnothing\} = \bigcup_{b \in B} X_{+b}$

Erosion (Minkowski subtraction): $X \ominus B = \{z : B_z \subseteq X\} = \bigcap_{b \in B} X_{-b}$

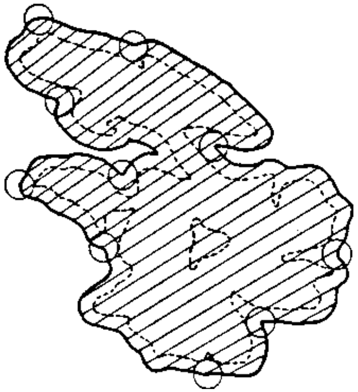
Minkowski Opening: $X \circ B = (X \ominus B) \oplus B$ **Closing:** $X \bullet B = (X \oplus B) \ominus B$



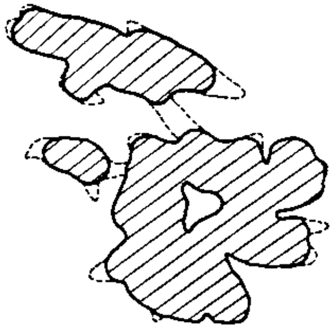
EROSION



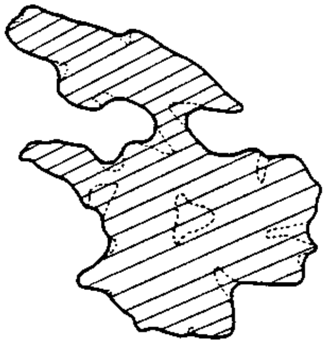
DILATION



OPENING



CLOSING



LEVEL SETS

- Image $f(x, y)$


IMAGE

- Level Sets (threshold sets): $X_h(f) = \{(x, y) : f(x, y) \geq h\}$


LEVEL SET 80


LEVEL SET 140


LEVEL SET 200

- Level Curves: $L_h(f) = \{(x, y) : f(x, y) = h\}$


LEVEL CURVE 80


LEVEL CURVE 140


LEVEL CURVE 200

- Image Reconstruction: $f(x, y) = \sup\{h : (x, y) \in X_h(f)\}$

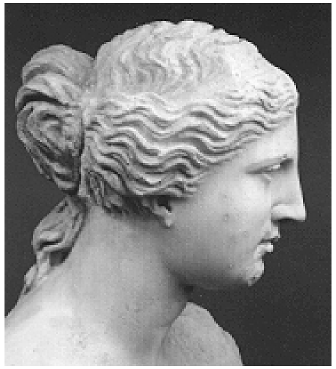
EUCLIDEAN MORPHOLOGICAL FLAT OPERATORS

Dilation: $(f \oplus B)(x) = \sup_{y\in B} f(x-y)$

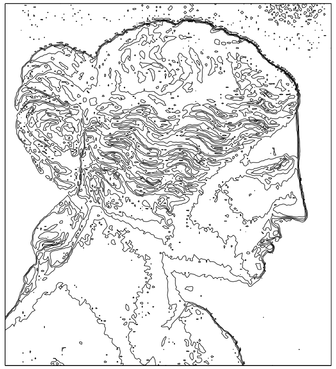
Erosion: $(f \ominus B)(x) = \inf_{y\in B} f(x+y)$

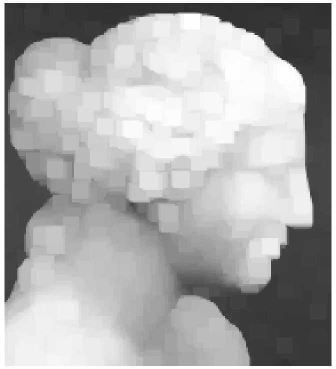
Opening: $f \circ B = (f \ominus B) \oplus B$

Closing: $f \bullet B = (f \oplus B) \ominus B$



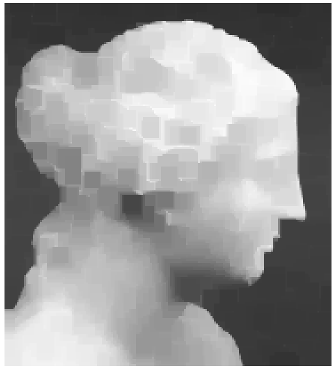
IMAGE

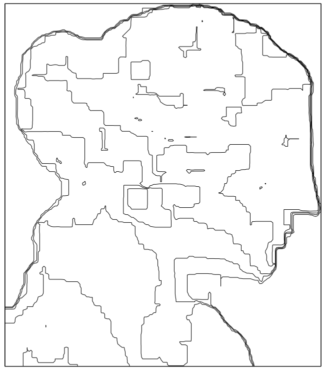

IM. LEVEL CURVES


DILATION 9x9


EROSION 9x9


OPENING 9x9


CLOSING 9x9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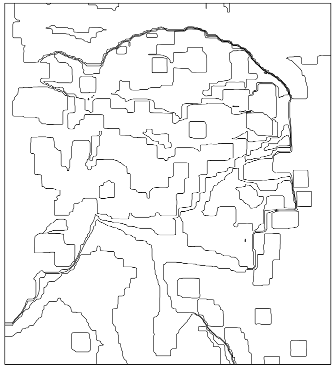
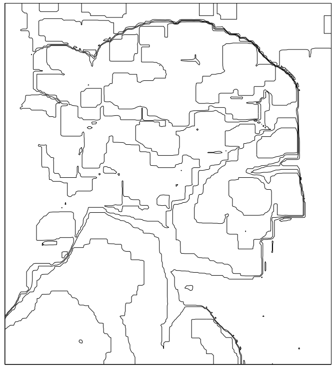
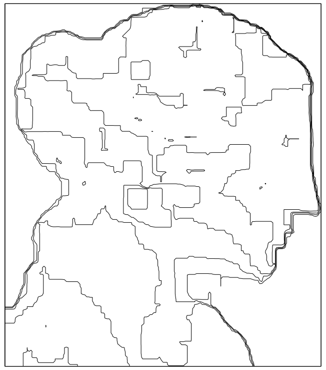

EROS. LEVEL CURVES


OPEN. LEVEL CURVES

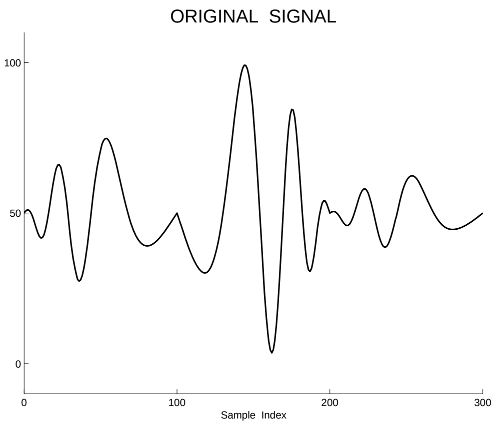

CLOS. LEVEL CURVES

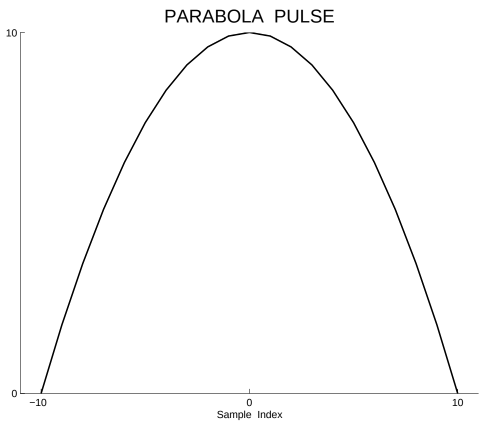
EUCLIDEAN MORPHOLOGICAL WEIGHTED OPERATORS

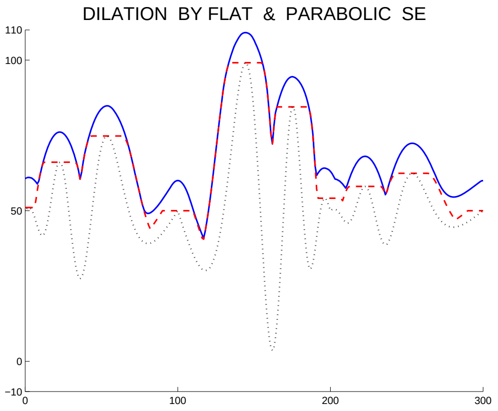
Dilation (Sup-plus convolution): $(f \oplus g)(x) = \vee_y f(y) + g(x - y)$

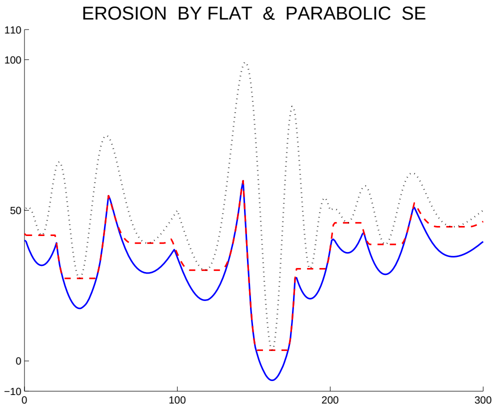
Erosion(Inf-plus correlation): $(f \ominus g)(x) = \wedge_y f(y) - g(y - x)$

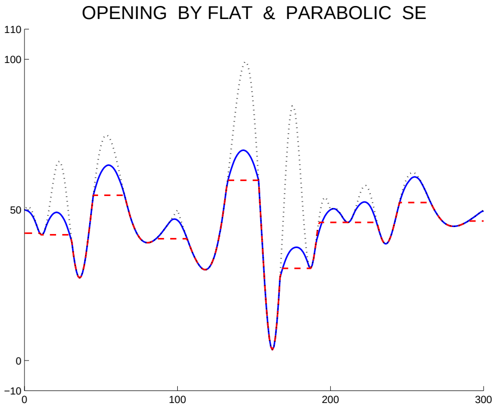
Opening: $f \circ g = (f \oplus g) \ominus g$ **Closing:** $f \bullet g = (f \oplus g) \ominus g$

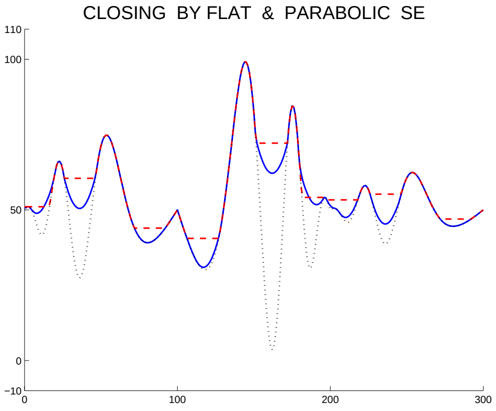

ORIGINAL SIGNAL


PARABOLA PULSE


DILATION BY FLAT & PARABOLIC SE


EROSION BY FLAT & PARABOLIC SE


OPENING BY FLAT & PARABOLIC SE


CLOSING BY FLAT & PARABOLIC SE

SUP- INF REPRESENTATION OF TI INCREASING OPERATORS

Theorem (Maragos 1985, IEEE T-PAMI 1989):

Every operator ψ on the set F of extended-real-valued functions that is *translation-invariant (TI) and monotone increasing* can be represented as a supremum (infimum) of Minkowski erosions (dilations) of the input by functions in its *kernel* K. If F consists of u.s.c. functions and the operator is also upper-semicontinuous, then ψ accepts a *basis* and the representation becomes *minimal*.

$$\boxed{\psi(f) = \mathrm{SUP}_{g \in K} f \ominus g = \mathrm{INF}_{h \in K^*} f \oplus h}$$

Applications:

- Composite Morphological operators
- Median, Rank, Stack filters
- Linear filters
- Image Denoising
- Curve Evolution, Curvature Motion

OPERATORS ON COMPLETE LATTICES

($\leq$ = partial ordering, $\vee$ = supremum, $\wedge$ = infimum)

- ψ is **increasing** iff $f \leq g \Rightarrow \psi(f) \leq \psi(g)$.
- δ is **dilation** iff $\delta(\vee_i f_i) = \vee_i \delta(f_i)$.
- ε is **erosion** iff $\varepsilon(\wedge_i f_i) = \wedge_i \varepsilon(f_i)$.
- α is **opening** iff increasing, antiextensive $(\alpha(f) \leq f)$,

  and idempotent $(\alpha = \alpha^2)$.
- β is **closing** iff increasing, extensive $(\beta(f) \geq f)$,

  and idempotent $(\beta = \beta^2)$.
- (ε, δ) is **adjunction** iff $\boxed{\delta(g) \leq f \Leftrightarrow g \leq \varepsilon(f).}$

  Then: ε is erosion, δ is dilation, $\varepsilon\delta$ is opening.

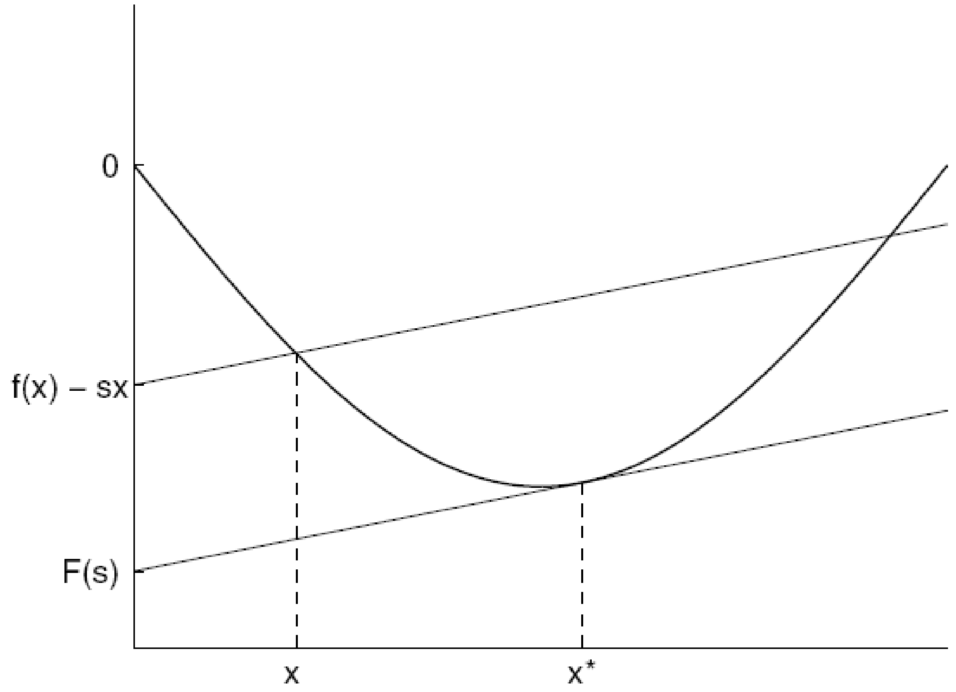
1980s: Matheron, Serra. 1990s: Heijmans, Ronse, Roerdink

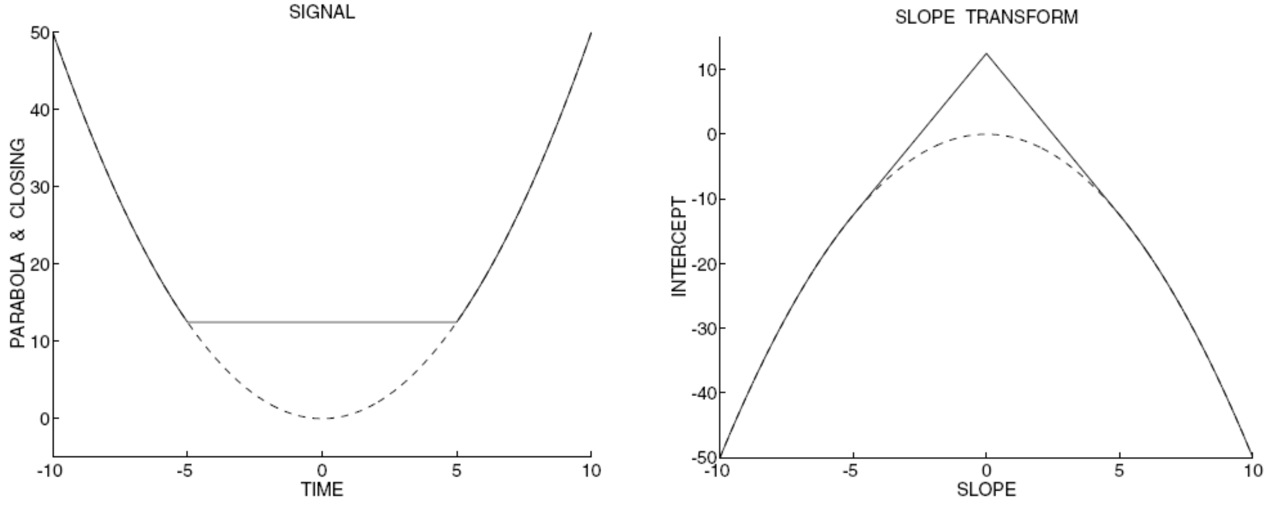
Slope Transform

Slope Transform

Upper: $F_{\vee}(\mathbf{s}) \triangleq \bigvee_{\mathbf{x}\in\mathbf{R}^2} f(\mathbf{x}) - \mathbf{s}\cdot\mathbf{x}$

Lower: $F_{\wedge}(\mathbf{s}) \triangleq \bigwedge_{\mathbf{x}\in\mathbf{R}^2} f(\mathbf{x}) - \mathbf{s}\cdot\mathbf{x}$





(a) (b)

Figure 10.3: (a) Convex parabola signal $f(x) = x^2/2$ (dashed line) and its morphological closing (solid line) by a flat structuring element $[-5, 5]$. (b) Lower slope transform $F_{\wedge}(s) = -s^2/2$ of the parabola (dashed line) and of its closing (solid line).

If f (x) is differentiable,
ST = Legendre transform.

LST is the negative of the Fenchel conjugate

"Inverse" Lower Slope Transform: $\check{f}(x) \triangleq \bigvee_{\mathbf{s}\in\mathbf{R}^2} F_{\wedge}(\mathbf{s}) + \mathbf{s}\cdot\mathbf{x}$

Slope Response

Zero Impulse, Zero Step

$$\xi(\mathbf{x}) \triangleq \begin{cases} 0, & \text{for } \mathbf{x} = \mathbf{0}, \\ -\infty, & \text{for } \mathbf{x} \neq \mathbf{0}, \end{cases} \qquad \zeta(\mathbf{x}) \triangleq \begin{cases} 0, & \text{for } \mathbf{x} \geq 0, \\ -\infty & \text{for } \mathbf{x} < 0, \end{cases}$$

DTI / ETI Systems in Space/Time domain as Sup/Inf convolutions

$$\mathcal{D} \text{ is DTI iff } \mathcal{D}(f) = f \oplus g_\vee, \text{ where } g_\vee \triangleq \mathcal{D}(\xi)$$

$$\mathcal{E} \text{ is ETI } \iff \mathcal{E}(f) = f \oplus' g_\wedge, \quad g_\wedge \triangleq \mathcal{E}(-\xi)$$

$f(\mathbf{x}) = \mathbf{s} \cdot \mathbf{x} + c$ are eigenfunctions of any DTI system $\mathcal{D}$ or ETI system $\mathcal{E}$

$$\mathcal{D}[\mathbf{s} \cdot \mathbf{x} + c] = \mathbf{s} \cdot \mathbf{x} + c + G_\vee(\mathbf{s})$$

$$\mathcal{E}[\mathbf{s} \cdot \mathbf{x} + c] = \mathbf{s} \cdot \mathbf{x} + c + G_\wedge(\mathbf{s})$$

DTI, ETI Systems in **Slope domain**: Upper/Lower **Slope Response**

$$\boxed{G_\vee(\mathbf{s}) \triangleq \bigvee_{\mathbf{x}} g_\vee(\mathbf{x}) - \mathbf{s} \cdot \mathbf{x}, \qquad G_\wedge(\mathbf{s}) \triangleq \bigwedge_{\mathbf{x}} g_\wedge(\mathbf{x}) - \mathbf{s} \cdot \mathbf{x}}$$

Sup/Inf Convolutions in Space become Additions in Slope domain

Maragos, IEEE T-SP 1995, T-IP 1996

Distance Transforms as Slope Filters

Sequential Distance Transform

$$u(i,j) = \left(\bigwedge_{(k,\ell)\in M_o} a_{k\ell} + u(i-k, j-\ell) \right) \wedge f(i,j)$$

Lower Slope Response:

$$G(\mathbf{s}) = \begin{cases} 0 & \text{for } \mathbf{s} \in P, \\ -\infty & \text{for } \mathbf{s} \notin P, \end{cases}$$

$$P = \{(s_1, s_2) : is_1 + js_2 \leq a_{ij} \ \forall (i,j) \in M_o\}$$

(a) (b) (c)

Fig. 5. Regions of support of binary slope responses of discrete ETI systems representing (a) the forward pass of the cityblock distance transform; (b) the forward pass of the chamfer (3, 4) distance transform; (c) the chamfer (3, 4) distance transform.

Maragos, T-IP 1996

Slope Limiting and Lipschitz Regularization

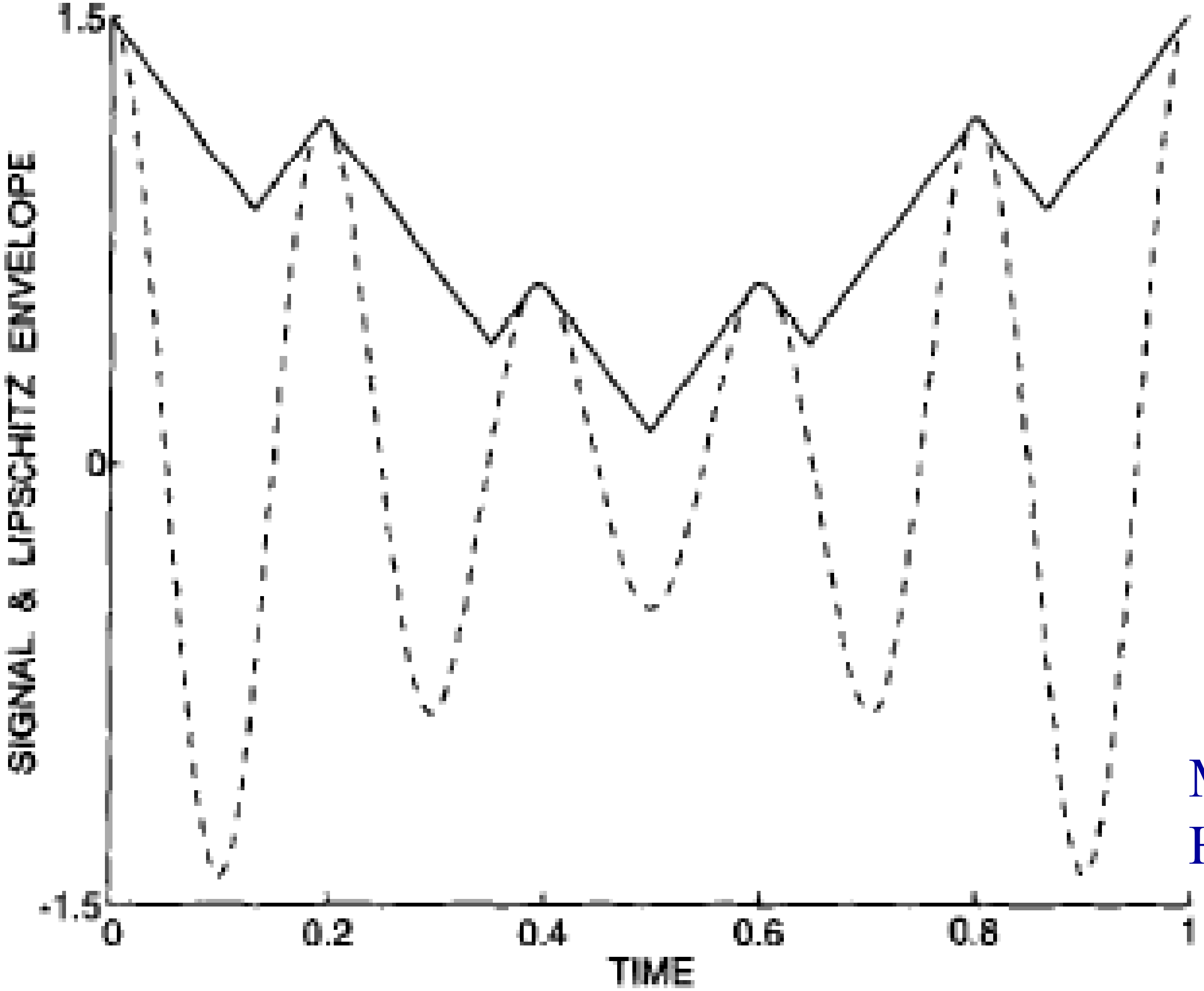


Fig. 3. Slope-limiting (i.e., Lipschitz regularization) of a function via its supremal convolution with a cone. The dashed line shows the original signal $f(x) = [1 + 0.5\cos(2\pi x)]\cos(10\pi x)$, $x \in [0, 1]$. The solid line is the supremal convolution of f with $K_a(x) = -a|x|$ where $a = -5$.

Maragos, SP 1994
Heijmans & Maragos, SP 1997

Connected Image Operators

Definition (flat zones): The set of flat zones of a graylevel function f is the set the connected components of the space where f is constant.

Definition (connected operators): A graylevel operator ψ is connected if the partition of flat zones of its input f is always finer than the partition of the flat zones of its output.

Properties of connected operators:

- If ψ is a connected operator, its dual $\psi^{*}(f)=-\psi(-f)$ is also connected
- If ψ_1, ψ_2 are connected operators, $\psi_1\psi_2$ is also connected
- If $\{\psi_i\}$ are connected, their supremum $\vee_i$ and infimum $\wedge_i$ are connected

Connected Operators:

- ✓ Act by merging flat zones
- ✓ Don’t create new image structures or new contours
- ✓ Have excellent preservation properties (don’t modify region boundaries)

Salembier & Serra, IEEE T-IP 1995

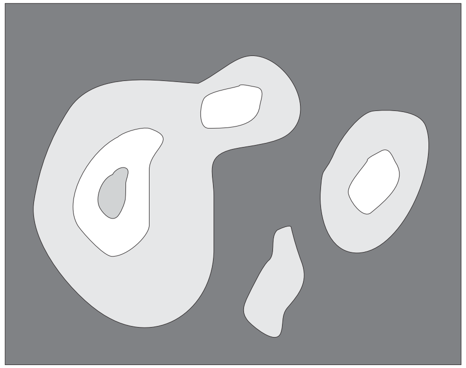
IMAGE SIMPLIFICATION

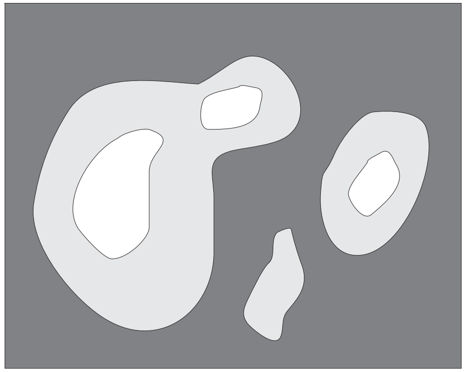
- Noise Reduction
- Structure Simplification
- Redundant Information Removal
- Preservation of Geometrical Structure and Objects' Contours

Tool: Connected Operators

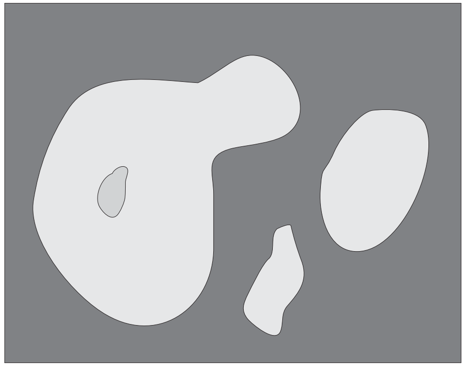
Properties:

- Merging connected components and flat zones
- Preservation of geometrical structure and objects' contours
- No introduction of new contours





Elimination of dark components



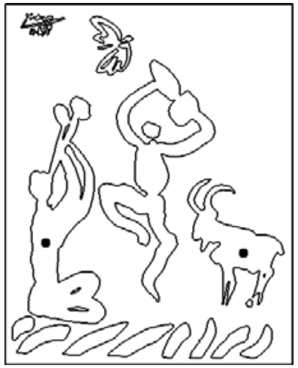
Elimination of bright components

CONTRAST FILTERING - Connected Operators Based on Reconstruction

Set Reconstruction (opening)

$\rho^-(M|X) =$ Connected component of X that includes M $= \lim_{n\to\infty}\left(\delta_B(\ldots\delta_B(\delta_B(M|X)|X)|X)\right)$


Binary Image


Markers


60 iterations


120 iterations


Final Result

Reconstruction Closing

$$\rho^+(m|f) = \lim_{n\to\infty} \varepsilon_B^n(m|f)$$

$$\varepsilon_B(m|f) = (m \ominus B) \vee f$$

Reconstruction Opening

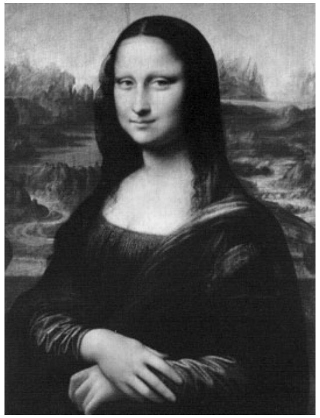
$$\rho^-(m|f) = \lim_{n\to\infty} \delta_B^n(m|f)$$

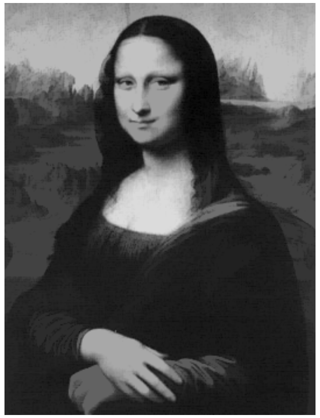
$$\delta_B(m|f) = (m \oplus B) \wedge f$$


Greyscale image *f*


Reconstruction Opening
(m=*f* - 40)


Reconstruction Closing
(m=*f* + 40)





AREA FILTERING - Connected Operators based on Area

Binary Area Opening

$$\alpha_n^- = \bigcup_i \{X_i : \text{Area}(X_i) \ge n\}$$

Binary Area Closing

$$\alpha_n^+(X) = [\alpha_n^-(X^c)]^c$$

Upper Level Sets

$$X_\vartheta(f) = \{(x,y) : f(x,y) \ge \vartheta\}$$

Binary Image

Area Opening, n=200

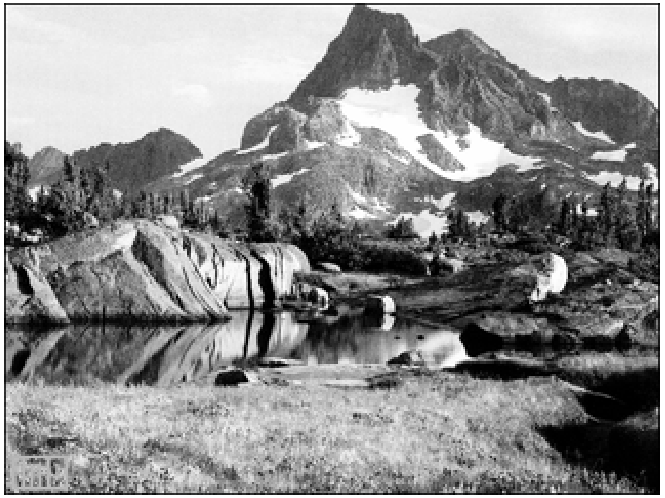
Area Opening n=1200

Greyscale Area Opening

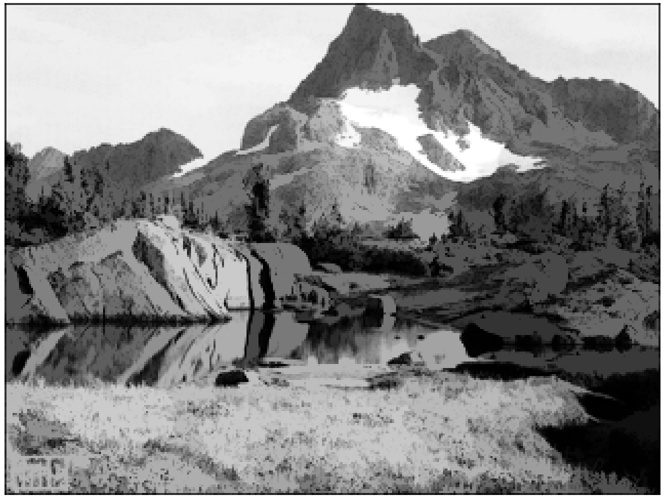
$$\alpha_n^-(f)(x,y) = \sup\{\vartheta : (\text{x,y}) \in \alpha_n^-(X_\vartheta(f))\}$$

Greyscale Area Closing

$$\alpha_n^+(f) = \sup\{\vartheta : (x,y) \in \alpha_n^+(X_\vartheta(f))\}$$



Greyscale Image



Area Opening

Area Closing

VOLUME FILTERING - Connected Operators Based On Volume

$$X_\vartheta(f) = \bigcup_i X_i \text{ και } Y = (X_\vartheta(f))^c = \bigcup_j Y_j$$

Upper Level Set Volume Opening

$$\beta_n^-(X) = \{X_i : \text{Area}(X_i)\cdot\vartheta \geq n\}$$

Upper Level Set Volume Closing

$$\beta_n^+(Y) = \{Y_j : \text{Area}(Y_j)\cdot\vartheta \geq n\}$$

Grayscale Volume Opening

$$\beta_n^-(f)(x,y) = \sup\{\vartheta : (x,y) \in \beta_n^-(X_\vartheta(f))\}$$

Grayscale Volume Closing

$$\beta_n^+(f)(x,y) = \sup\{\vartheta : (x,y) \in \beta_n^+(X_\vartheta(f))\}$$

Grayscale Image

Area Opening

Volume Opening

Area Closing

Volume Closing

LEVELINGS - Self Dual Filtering

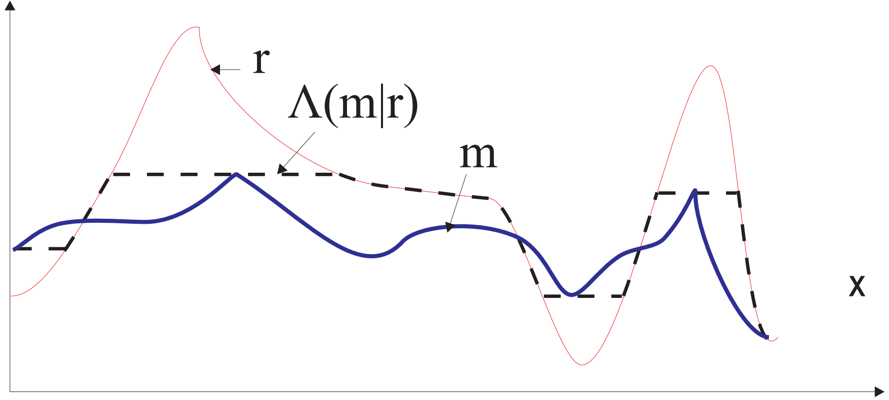
Self Dual Filtering: Symmetrical treatment of bright and dark image components

Leveling

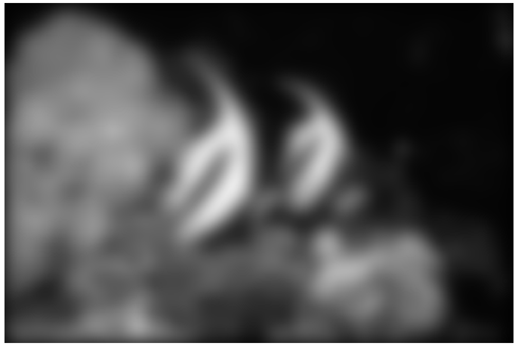
$$\Lambda(m \mid r) = \lim_{k \to \infty} f_k, \;\; f_k = \lambda(f_{k-1} \mid r), \;\; f_0 = m$$

$$\lambda(f \mid r) = (\delta(f) \wedge r) \vee \varepsilon(f)$$

δ dilation, ε erosion,with disk B




Image


marker m


Leveling

Alternating Sequential Filtering

$$\Psi_{\mathrm{ASF}}(f) = \varphi_n(\gamma_n(\ldots(\varphi_2(\gamma_2(\varphi_1(\gamma_1(f))))\ldots))$$

φ closing, γ opening

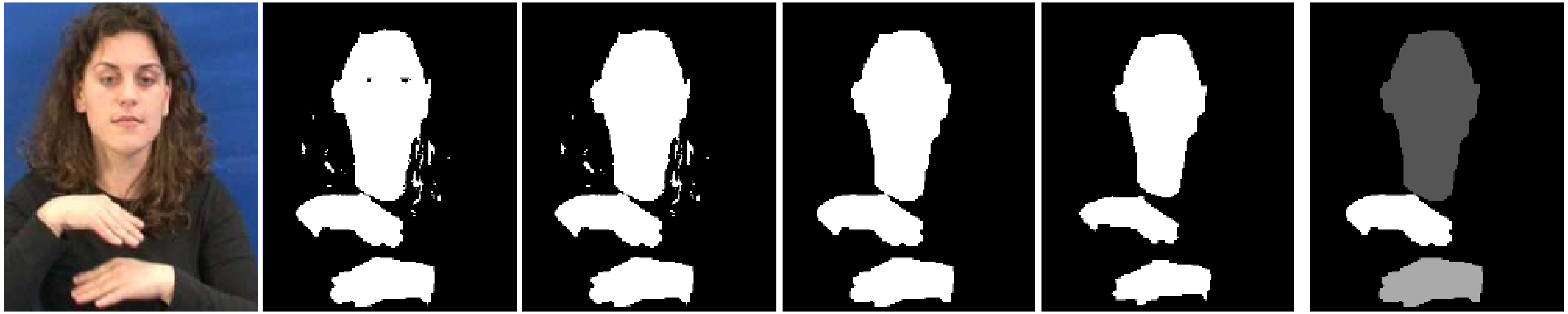

Image f


$\Psi_{ASF}(f)$, n=6

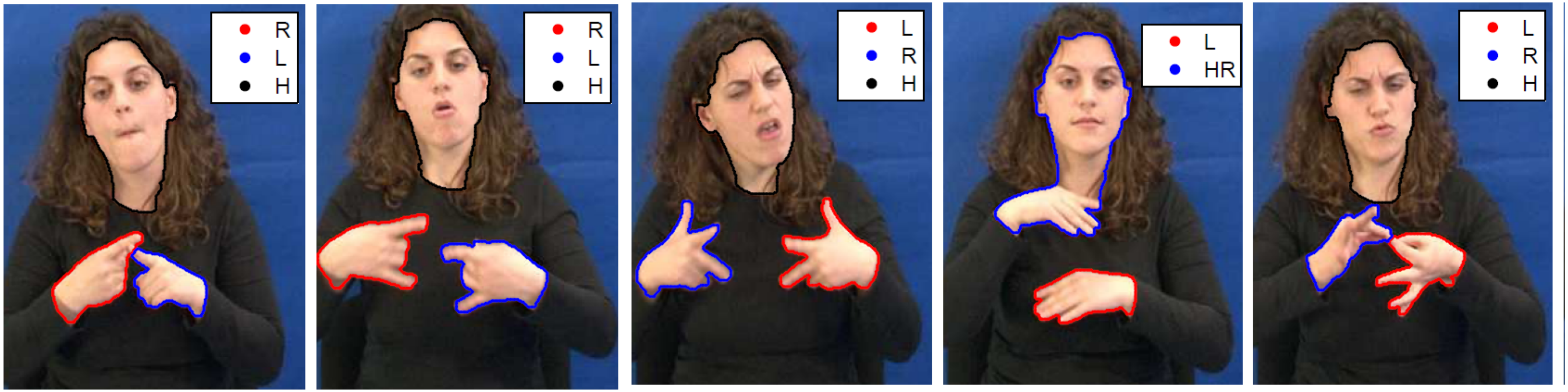

$\Psi_{ASF}(f)$, n=10

Tracking and Motion Feature Extraction in Sign Language Recognition

- Skin mask extraction, morphological segmentation



(a) Input (b) S_0 (c) S_1 (d) S_2 (e) $S_2 \ominus B$ (f) Segmented S_2



- **Hands and Head Tracking and labeling**

Morphological PDEs:
Nonlinear Scale-Spaces
and
Variational formulation

PREVIOUS WORK RELATED

- **Erosion – Dilation PDEs:**

Infinites. Gener. of Oper. Semigroups: Brockett & Maragos 1992 - 94.

Scale-Space Axiomatics: Alvarez, Guichard, Lions & Morel: 1992-93

Propagation of Boundaries: Boomgaard & Smeulders: 1992, 1994.

- **Curve Evolution Implementation of Continuous-Scale MM:**

Arehart, Vincent & Kimia 1993

Sapiro, Kimmel et al. 1993

- **Levelings:**

Leveling-based Scale-Spaces [Meyer & Maragos 1999]

PDE for Levelings: [Maragos & Meyer 1999]

Inf-Semilattice & Self-DualMorhology:[Keshet, Heijmans 1998, 2000]

Scale-Spaces of Triphase Operators → Leveling PDE: [Maragos, IJCV 2003]

GAUSSIAN SCALE SPACE AND HEAT PDE

- initial image $f(x, y)$

- multiscale Gaussian convolutions: $u(x, y, t) = f(x, y) * G_t(x, y)$

$t = 4$

$t = 8$

$t = 32$

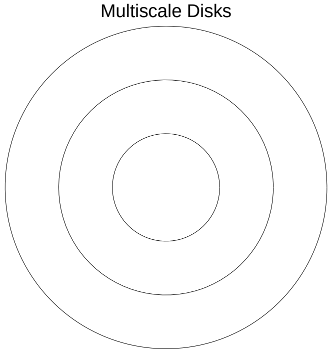
- 2D (isotropic) heat diffusion PDE:

initial condition: $u(x, y, 0) = f(x, y)$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u}$$

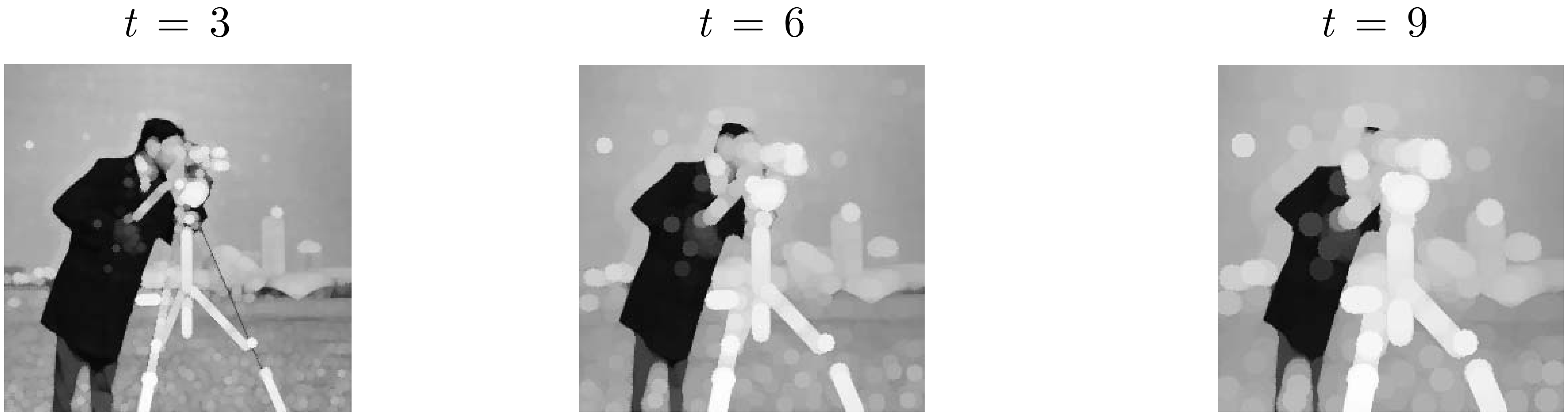
PDE FOR 2D MULTISCALE FLAT DILATIONS

- initial image $f(x,y)$, multiscale flat struct. elements



tB

- multiscale dilations by disks : $\delta(x,y,t) = (f \oplus tB)(x,y)$



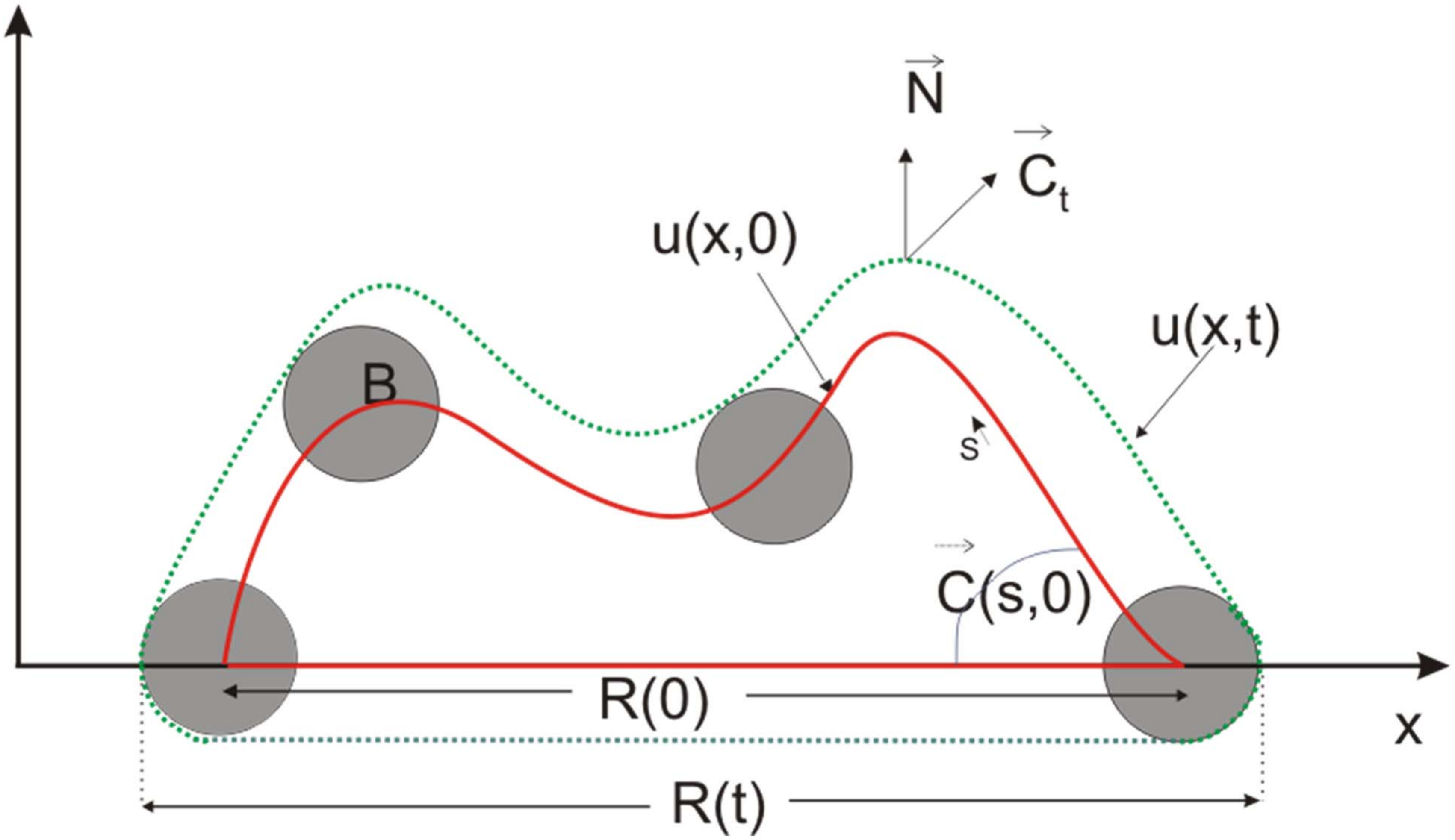
- PDE

$$\frac{\partial \delta}{\partial t} = \left\| \nabla \delta \right\|_2 = \sqrt{\left(\frac{\partial \delta}{\partial x}\right)^2 + \left(\frac{\partial \delta}{\partial y}\right)^2}$$

$$\boxed{B = \ell_p \text{ ball} \;\Rightarrow\; \frac{\partial \delta}{\partial t} = \left\| \nabla \delta \right\|_q}$$

Brockett & Maragos 1992, IEEE T-SP 94

Function Evolution via Hypograph Dilation



$$\left.\frac{d}{dt}A(t)\right|_{t=0} \propto \int \left\langle \vec{C}_t, \vec{N} \right\rangle ds \ + \text{const.}$$

Volume Extremization with Sup-Inf Constraints

Theorem : Maximizing the volume functional by keeping invariant the global supremum

$$\max \int\int u\,dxdy \quad \text{s.t.} \quad \bigvee u = \bigvee u_0$$

has a gradient flow governed by the **PDE generating flat dilation by disks**

$$u_t = \|\nabla u\|, \qquad u(x,y,0) = u_0(x,y)$$

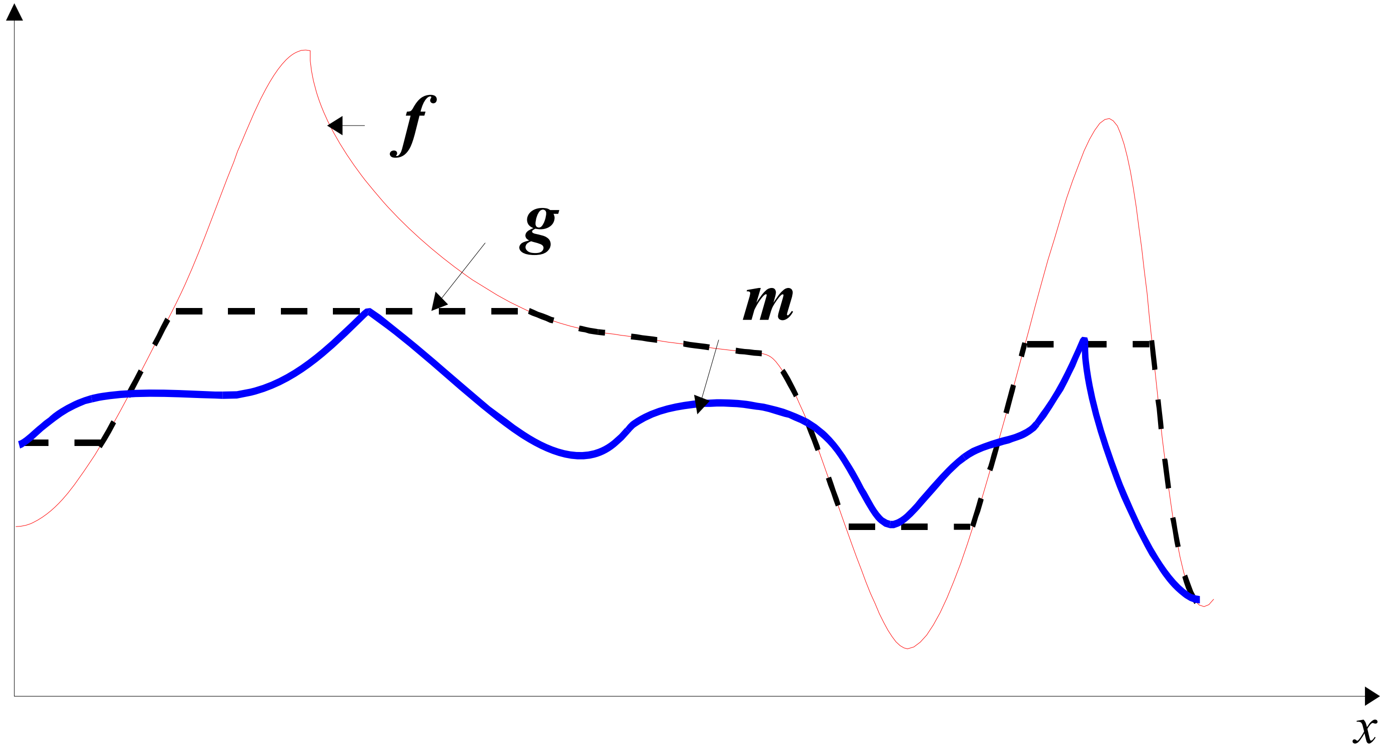
Similarly, the **dual problem** of minimizing the volume functional by keeping invariant the global infimum

$$\min \iint u\,dxdy \qquad \text{s.t.} \quad \bigwedge u = \bigwedge u_0$$

has a gradient vector flow governed by the **isotropic flat erosion PDE**:

$$u_t = -\|\nabla u\|, \qquad u(x,y,0) = u_0(x,y)$$

Construction of a Leveling associated to a Reference function *f* and a Marker *m*



Meyer ISMM-1998

On $\{x: m(x)<f(x)\}$, the leveling increases *m* until creating a flat zone or hitting *f* : hence on $\{x:g(x)<f(x)\}$, the function *g* is flat.
On $\{x: m(x)>f(x)\}$, the leveling decreases *m* until creating a flat zone or hitting *f* : hence on $\{x: m(x)>f(x)\}$, the function *g* is flat.

Discrete Algorithm: $g = \lim_{k\to\infty} g_k, \quad g_{k+1} = (\delta_B(g_k) \wedge f) \vee \varepsilon_B(g_k), \quad g_0 = m.$

LEVELINGS: DEFINITION via TRIPHASE OPERATORS

Triphase: $\lambda(m \mid f) = \alpha(m \mid \beta(m \mid f)) = \beta(m \mid \alpha(m \mid f))$

α, β are increasing, α is antiextensive, β is antiextensive

(α, β) is duality $\Rightarrow \lambda$ is self-dual

- A signal g is defined as a λ-**Leveling** of f if it is a fixed point of the triphase operator λ, i.e : $g = \lambda\ (g \mid f)$
- Levelings can be obtained as inf-semilattice infimum of iterated self-compositions of triphase operators:

$$\Lambda(m \mid f) = \lambda^{\infty}(m \mid f) \preceq_f \ldots \preceq_f \ldots \preceq_f \lambda^2(m) \preceq_f \lambda(m) \preceq_f m$$

- Leveling operator $f \mapsto \Lambda(m \mid f)$ is increasing and idempotent .

MULTISCALE OPERATORS ON COMPLETE LATTICES

- **Minkowski** multiscale dilation/erosion by disks tB at scales $t \geq 0$:

$$\delta_B^t(f) = f \oplus tB, \qquad \varepsilon_{\mathrm{B}}^t(f) = f \ominus tB$$

- **Conditional** multiscale dilation/erosion:

$$\delta_{tB}(m \mid f) = (m \oplus tB) \wedge f, \quad \varepsilon_{tB}(m \mid f) = (m \ominus tB) \vee f$$

- **Geodesic** multiscale dilation/erosion:

$$\delta^t(m \mid f)(x) = \{v : x \in \delta^t(X_v(m) \mid X_v(f))\}, \quad X_v(.) = \text{ Level Sets}$$

$$\varepsilon^t(m \mid f) = -\delta^t(-m \mid -f)$$

- Additive **Semigroups** :

$$\text{Unconstrained:} \quad \delta_B^t \delta_B^s(f) = \delta_B^{s+t}(f)$$

$$\text{Conditional:} \quad \delta_{tB}(\delta_{sB}(m \mid f) \mid f) \neq \delta_{(s+t)B}(m \mid f)$$

$$\text{Geodesic:} \quad \delta^t(\delta^s(m \mid f) \mid f) = \delta^{s+t}(m \mid f)$$

CONSTRAINED MULTISCALE OPERATOR

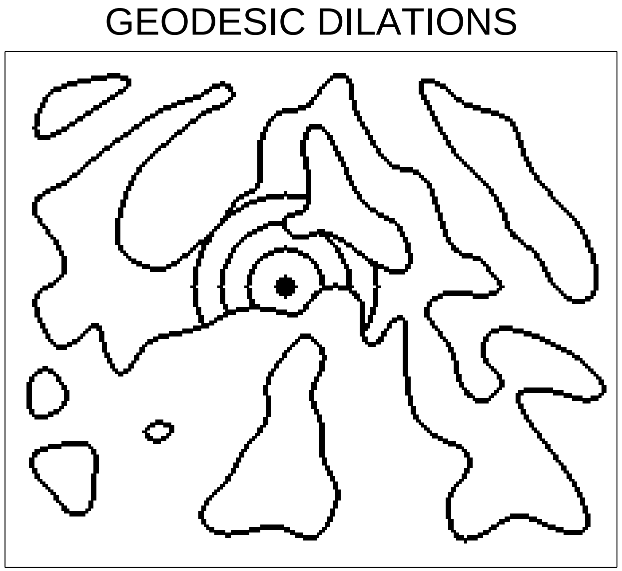
- **Geodesic multiscale dilation** $\delta^t(.|.)$

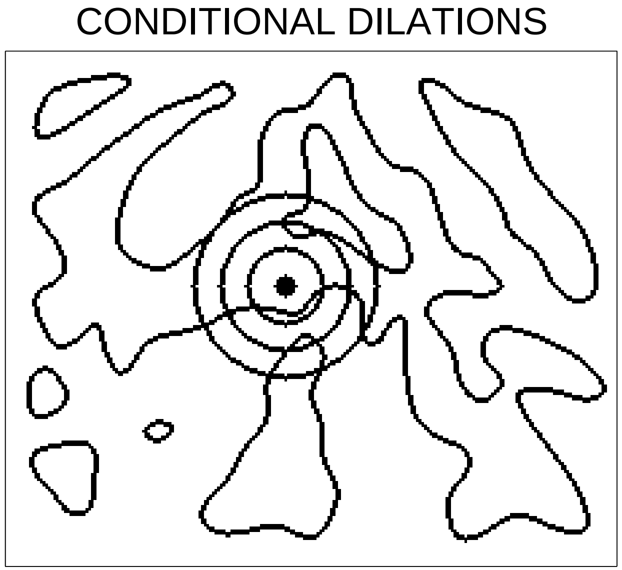
Binary images: $\delta^t(X \mid R) = \{x : d_R(x,p) \le t, \text{ some } p \in X\}$

X = marker, R = reference set, d_R=geodesic metric of R

Graylevel images: $\delta^t(m \mid f)$ is defined from threshold superposition of

$\delta^t(X \mid R)$ acting on level sets of m and f.





MULTISCALE TRIPHASE OPERATORS → LEVELINGS

- **Conditional Triphase:**

$$\lambda_{tB}(m\,|\,f)=[f\wedge\delta_B^t(m)]\vee\varepsilon_B^t(m)=\varepsilon_{tB}(m\,|\,\delta_{tB}(m\,|\,f))$$

- **Geodesic Triphase:**

$$\lambda^t(m\,|\,f)=\varepsilon^t(m\,|\,\delta^t(m\,|\,f))=\delta^t(m\,|\,\varepsilon^t(m\,|\,f))$$

- **Additive Semigroup**

$$\text{geodesic triphase: }\ \lambda^t(\lambda^s(m\,|\,f)\,|\,f)=\lambda^{s+t}(m\,|\,f)$$

- **Geodesic Leveling:**

$$\Lambda(m\,|\,f)=\lambda^\infty(m\,|\,f)=\delta^\infty(m\,|\,\varepsilon^\infty(m\,|\,f))$$

- **Conditional Leveling:**

$$\Lambda_B(m\,|\,f)=\lambda_B^\infty(m\,|\,f)\ \preceq_f\ \Lambda(m\,|\,f)$$

PDE FOR 2D LEVELINGS

$f(x,y) =$ reference image, $\quad$ $\mathrm{m}(x,y) =$ marker image

- **Geodesic triphase scale-space**: $u(x,y,t) = \lambda^t(m \mid f)(x,y)$
- **PDE**

$$u_t(x,y,t) = -\mathrm{sign}[u(x,y,t) - f(x,y)] \|\nabla u(x,y,t)\|$$

$$u(x,y,0) = m(x,y)$$

- **Numerical Algorithm** (shock-capturing & entropy-satisfying for $\theta = \Delta t / \Delta x \leq 1$)

Grid: $U_{i,j}^n = \hat{u}(i\Delta x, j\Delta y, n\Delta t)$.

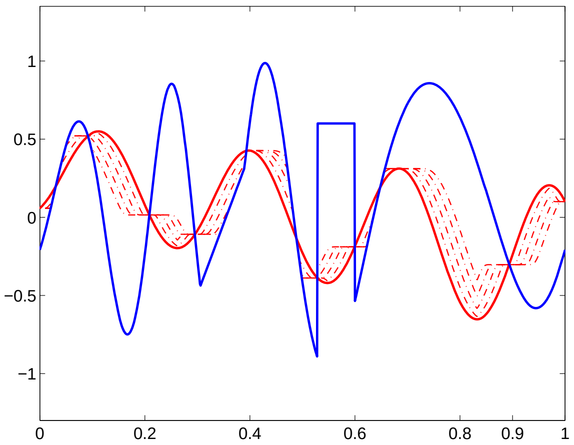
Conditional Triphase: $\quad \Phi(M_{i,j}) \equiv [F_{ij} \wedge \beta(M_{ij})] \vee \alpha(M_{ij})$

Iterate: $U_{i,j}^n = \Phi(U_{i,j}^{n-1}) = \Phi^n(U_{i,j}^0), \quad n = 1, 2, \ldots, \infty.$

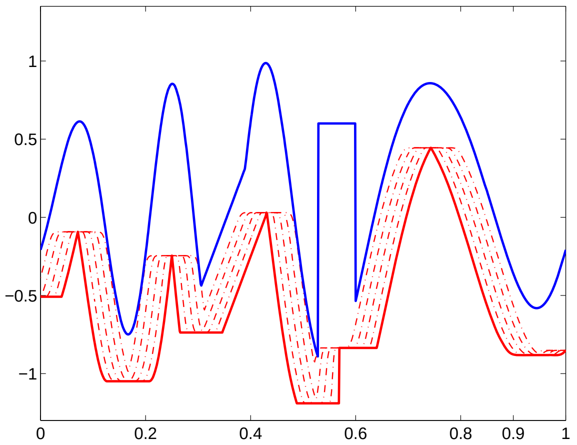
- **Convergence** to a unique leveling $U^\infty = \Phi^\infty(U^0)$.

Maragos & Meyer, SS 1999
Maragos, IJCV 2003

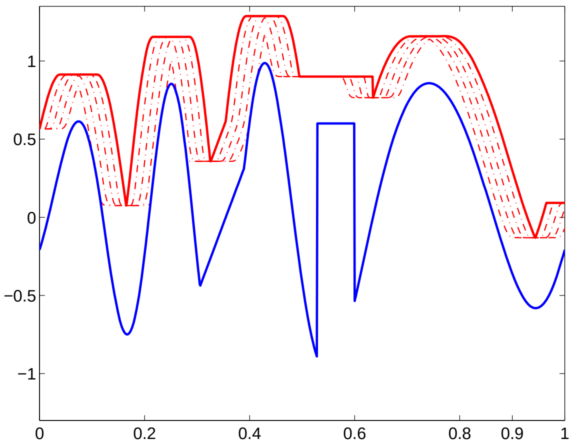
PDE – GENERATED 1D LEVELINGS



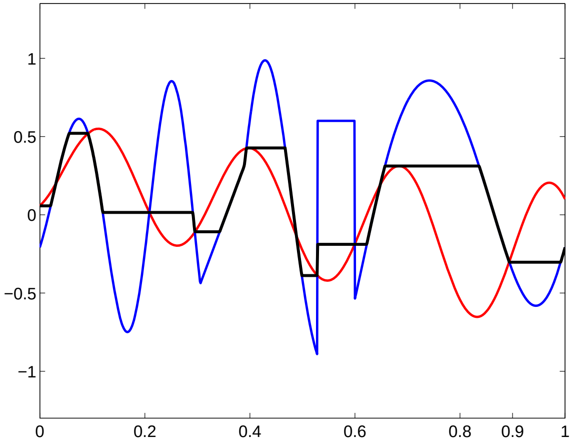
LEVELING PDE EVOLUTIONS



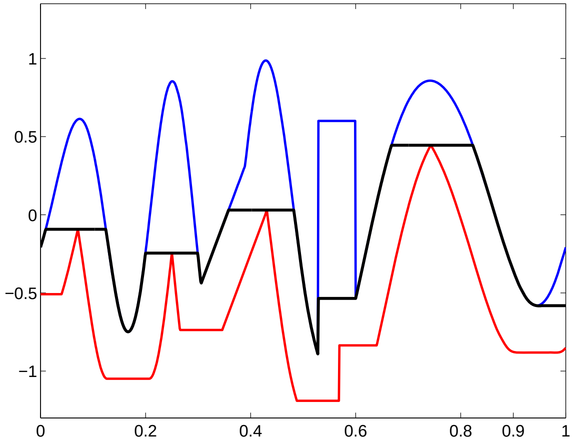
REC.OPEN. PDE EVOLUTIONS



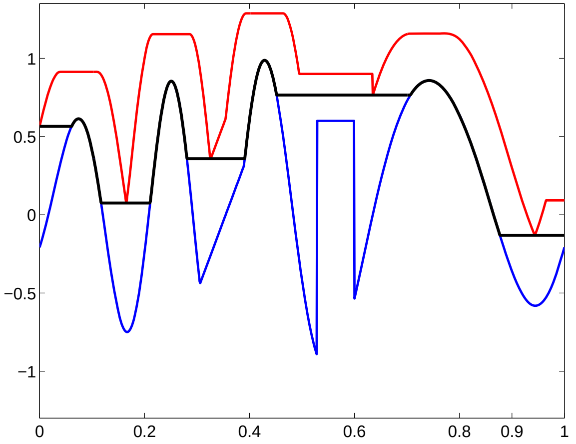
REC.CLOS. PDE EVOLUTIONS



REFERENCE, MARKER, & LEVELING



REFERENCE, MARKER, & REC.OPENING

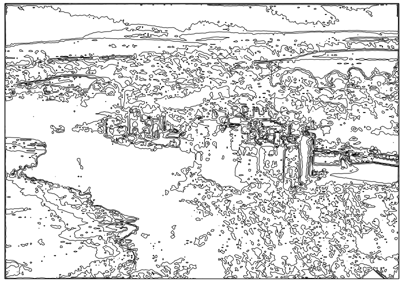


REFERENCE, MARKER, & REC.CLOSING

GAUSS & LEVELING SCALE - SPACE

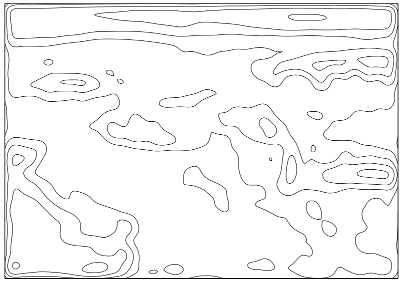
Reference

Reference: Level Curves



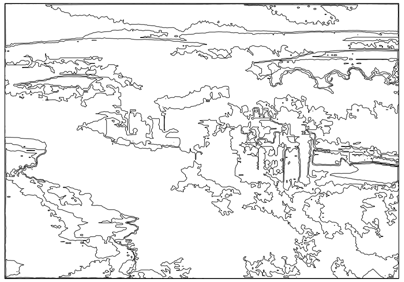
Gaus. Marker 1 (σ=4)

Marker 1: Level Curves



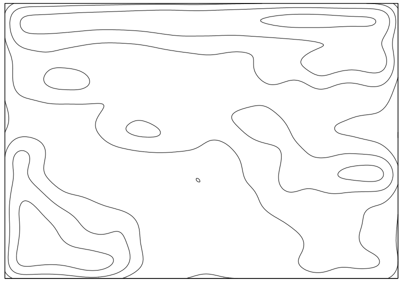
Leveling 1

Leveling 1: Level Curves



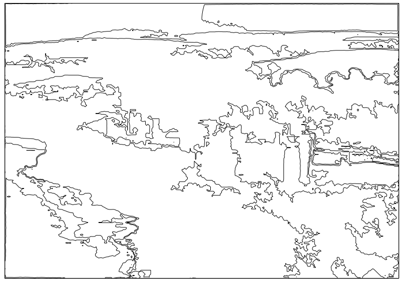
Gaus. Marker 2 (σ=8)

Marker 2: Level Curves



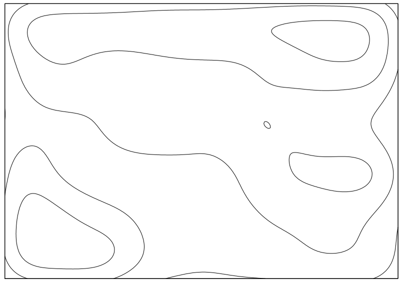
Leveling 2

Leveling 2: Level Curves



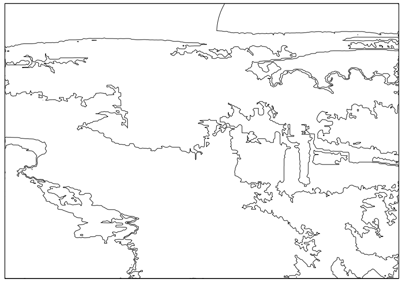
Gaus. Marker 3 (σ=16)

Marker 3: Level Curves



Leveling 3

Leveling 3: Level Curves



Meyer & Maragos, JVCIR 2000

Create a cartoon simplification of a reference image $f(x,y)$ consisting of several parts by using a marker $u_0(x,y)$ that intersects some of these parts and evolves towards f in a monotone way such that all evolutions $u(x,y,t)$ satisfy:

$$t_1 < t_2 \Rightarrow f(x,y) \preceq_f u(x,y,t_2) \preceq_f u(x,y,t_1) \preceq_f u_0(x,y)$$

$\preceq_f$ is a inf-semilattice order w.r.t. a reference f

$$\Rightarrow \left| f(x,y) - u(x,y,t) \right| \leq \left| f(x,y) - u_0(x,y) \right| \quad \forall t$$

Partition the regions R^- and R^+ formed by zero-crossings of $f - u_0$:

$$R^- = \{(x,y) : f(x,y) \geq u_0(x,y)\} = \sqcup_i R_i^-$$

$$R^+ = \{(x,y) : f(x,y) < u_0(x,y)\} = \sqcup_i R_i^+$$

Evolution of u is done by maintaining all local min/max of u_0 inside subregions R_i^- / R_i^+ :

$$\bigvee_{R_i^-} u = \bigvee_{R_i^-} u_0 \quad \text{and} \quad \bigwedge_{R_i^+} u = \bigwedge_{R_i^+} u_0,$$

Variational Formulation of Levelings

Theorem: The gradient flow for the optimization problem

$$\boxed{\min \iint |u-f|\,dxdy \quad \text{s.t.} \quad \bigvee_{R_i^-} u = \bigvee_{R_i^-} u_0, \quad \bigwedge_{R_i^+} u = \bigwedge_{R_i^+} u_0}$$

is given by:

$$\boxed{\begin{array}{l} \partial u(x,y,t)/\partial t = -\operatorname{sgn}(u-f)\,\|\nabla u\| \\ u(x,y,0) = u_0(x,y) \end{array}}$$

$$R^- = \{(x,y) : u_0(x,y) \geq f(x,y)\} = \sqcup_i R_i^-$$

$$R^+ = \{(x,y) : u_0(x,y) < f(x,y)\} = \sqcup_i R_i^+$$

Maragos, IJCV 2003

U+V Decomposition and related PDE-based Watershed & T-Energy Image Segmentations

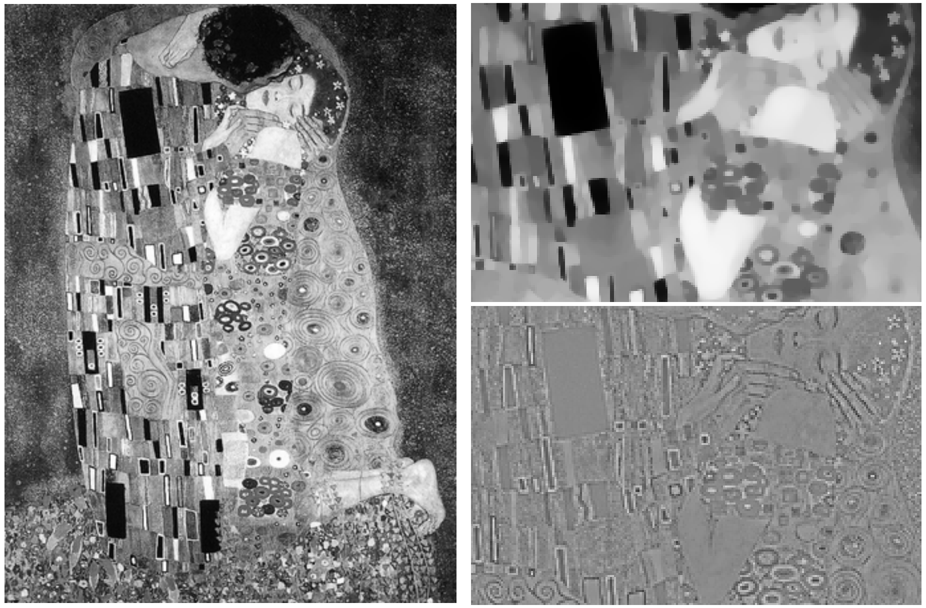
References:

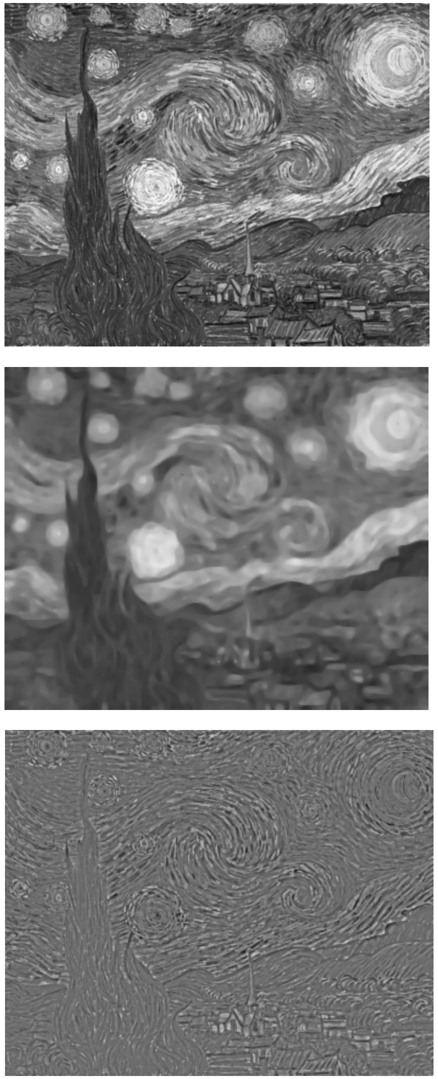
- P. Maragos & G. Evangelopoulos, ISMM-2007
- G. Evangelopoulos & P. Maragos, CVPR-2008.
- A. Sofou & P. Maragos, IEEE T-IP 20008

Image decomposition

- Image (u+v) model $f = u + v, \quad f, u, v : \Omega \subset \Re^2 \to \Re$
  - «cartoon» u (edges, contours, objects, shapes)
  - texture v (oscillations, details, noise)

- Inverse problem: image decomposition
  - Energy minimization
    - Total variation, convex optimization, PDE’s
    - Wavelets and projections on function bases, dictionaries
  - Applications: image restoration, inpainting, analysis





Variational schemes

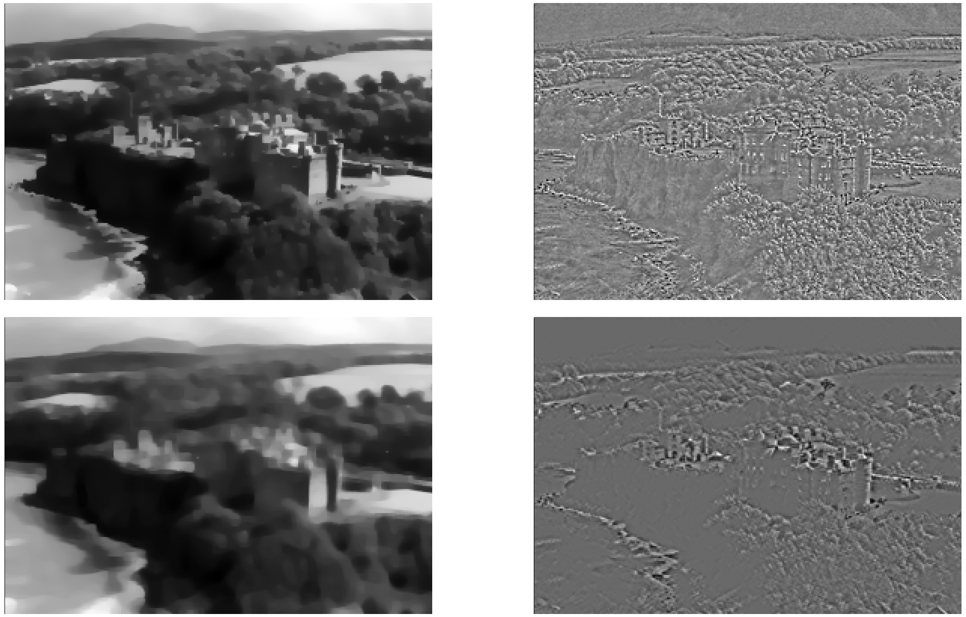
- *Mumford-Shah* image simplification
- Total Variation minimization (*Rudin, Osher & Fatemi):* $\|u\|_{\mathrm{TV}} = \iint_{\Omega} \|\nabla u\| dxdy$

$$E_{ROF}(u) = \|u\|_{\mathrm{TV}} + \lambda \|u - f\|_2^2$$

- Texture = Oscillatory functions *(Y. Meyer):* $v = \mathrm{div}\vec{g} = \partial_x g_1 + \partial_y g_2$
- u+v *(Vese & Osher):* $E_{VO}(u,\vec{g}) = \|u\|_{\mathrm{TV}} + \lambda \left\| f - (u + \mathrm{div}\vec{g}) \right\|_2^2 + \mu \|\vec{g}\|_p$

image

cartoon *texture*



Leveling-based cartoons

- Leveling cartoon approximations $u = \Lambda(M \mid f)$
  - u: leveling of image f
  - M: marker (e.g. Gaussian, anisotropic)

- Residual $r = f - u$
  - finer scales information
  - *contains* texture v

- Multi-scale levelings
  - hierarchy of cartoons/residuals

$$\boxed{u_i = \Lambda(M_i \mid u_{i-1}), i = 1, 2, \ldots, n}$$

$$u_0 = f \qquad \boxed{r_i = f - u_i}$$

  - causality property: *u_j is a leveling of u_i for j>i*
  - markers are samples of a scale-space $M_i = f * G_{\sigma_i}$

level 0 (image)

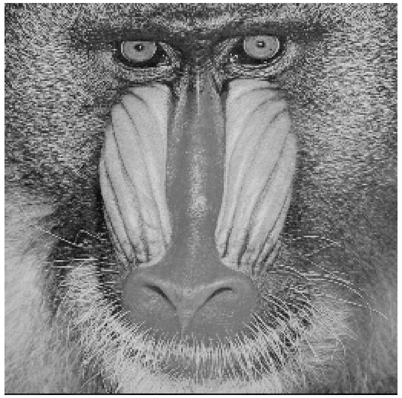
level 1 (σ1=4)

level 2 (σ2=16)

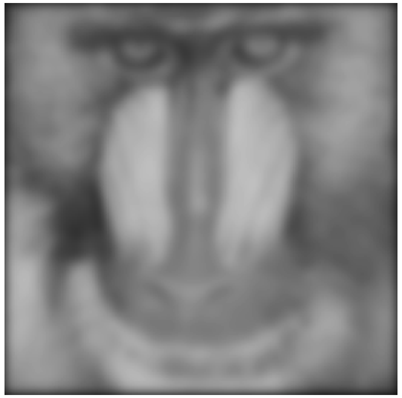
Multiscale leveling decomposition (example)

- 2-level cartoons and residuals via levelings with markers-samples of an isotropic Gaussian scale-space

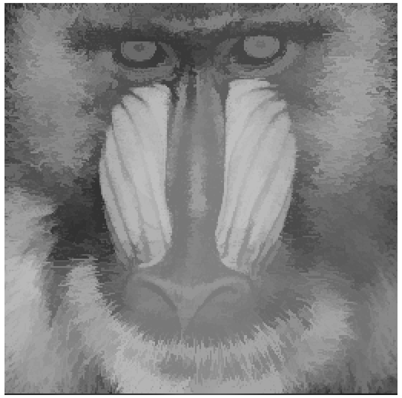
image *marker* *leveling* *residual*



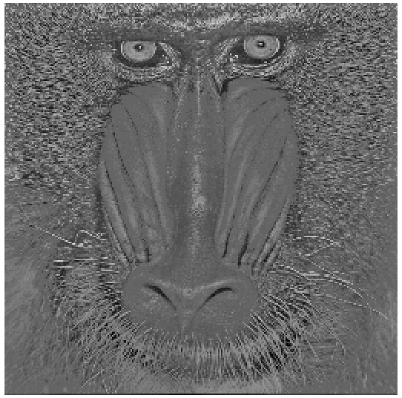
f



$M_1 = f * G_{\sigma 1}$



$u_1 = \Lambda\left(M_1 \mid f\right)$



$r_1 = f - u_1$

level 1,
(σ1=4)

$u_1 - u_2$

$M_2 = f * G_{\sigma 2}$

$u_2 = \Lambda\left(M_2 \mid u_1\right)$

$r_2 = f - u_2$

level 2,
(σ2=8)

Comparisons with TV cartoons

- **Levelings decrease the Total Variation norm**

$$\iint \|\nabla u_{i+1}\| \leq \iint \|\nabla u_i\| \leq \iint \|\nabla f\|$$

Leveling cartoons

a. preserve regional maxima & minima and do not create new

b. preserve the sense of variation between neighbour pixels

c. TV norm decreases monotonically

d. scale controlled by (the scale) of the marker image

TV cartoons

a. preserve the global mean

b. preserve the global variance

c. scale controlled by the regularizing constant

Levelings for Smoothing-Decomposition

ROF (TVmin)

OSV (U+V)

Leveling (Gauss)

Leveling (Anis.)

 # iter = 50

$|U|_{TV}$ =10.7

 # iter = 50

$|U|_{TV}$ =10.2

σ = 4

$|U|_{TV}$ =13.4

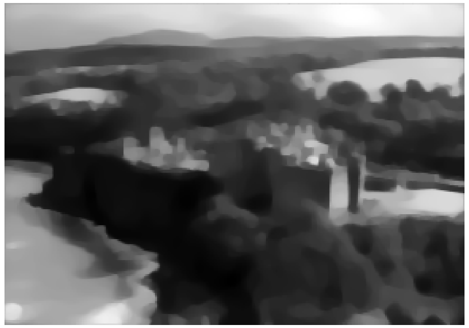
K = 20

$|U|_{TV}$ =16.3

 # iter = 100

$|U|_{TV}$ =8

 # iter = 100



$|U|_{TV}$ =9.1

σ = 16

$|U|_{TV}$ =8.4

K = 40

$|U|_{TV}$ =14.5

AM-FM Texture Model

- Locally narrowband image texture *(Bovik, Havlicek etal.)*

$$f(x,y) = a(x,y)\cdot\cos\left[\phi(x,y)\right], \qquad \nabla\phi(x,y) = \vec{\omega}(x,y)$$

  - analogies between AM-FM and Y.Meyer’s oscillating functions for texture

- Inst. Amplitude & Frequency estimation *(Maragos & Bovik , JOSA1995):*
  - ❑ Multiband Gabor filtering
  - ❑ **2D Energy Operator** $\boxed{\Psi(f) = \|\nabla f\|^2 - f\nabla^2 f}$
  - ❑ Demodulation via the **Energy Separation Algorithm (ESA):**

$$\frac{\Psi(f)}{\sqrt{\Psi(\partial f/\partial x) + \Psi(\partial f/\partial y)}} \approx |a(x,y)|$$

$$\sqrt{\Psi(\partial f/\partial x)/\Psi(f)} \approx |\omega_1(x,y)|, \qquad \sqrt{\Psi(\partial f/\partial y)/\Psi(f)} \approx |\omega_2(x,y)|$$

Multiband Texture Energy Tracking (I)

- *Texture modulation energy* of a locally narrowband component

$$f(x,y) = a(x,y)\cdot\cos\left[\phi(x,y)\right], \qquad \Psi\left[a\cos(\phi)\right] \approx a^2 \left\|\nabla\phi\right\|^2$$

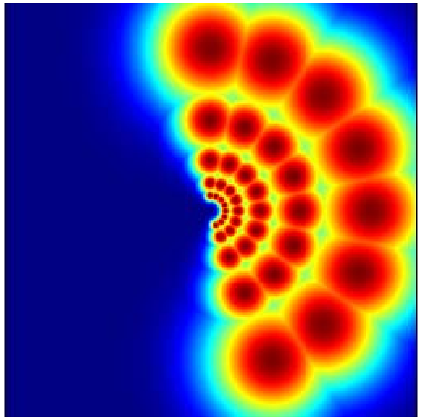
- Bandpass filter the "texture image part" to isolate components

$$f_k(x,y) = (v * g_k)(x,y) \approx (v_k * g_k)(x,y),$$

- Impulse Responses of a 2D Gabor filterbank

$$h_k(x,y) = \exp\left\{-a_k^2x^2 - b_k^2y^2\right\}\exp\left\{j\vec{\Omega}_k\cdot(x,y)\right\}$$

- k-filter:
  - ☐ Bandwidth parameters (a_k, b_k),
  - ☐ central frequency vector $\vec{\Omega}_k$



- Filterbank design
  - ☐ polar arrangement in spectral domain
  - ☐ octave bandwidth, equal bandwidth params
  - ☐ typical design (40 filters, 5 scales, 8 orientations)

Multiband Texture Energy Tracking (II)

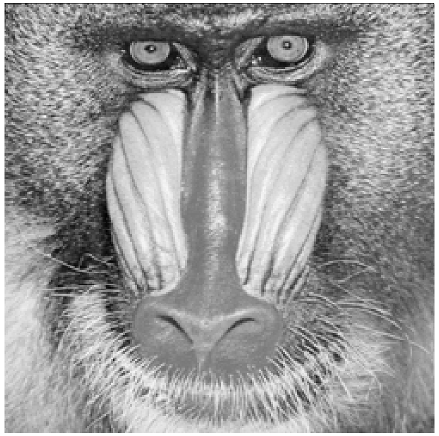
- Maximum Average Teager *(MAT)* energy
  - Energy tracking from the set of filtered, narrowband texture components

$$\Psi_{\mathrm{mat}}\left(v(x,y)\right)=\arg\max_{k}\left\{\left(\Psi\left(v*h_k\right)*h_a\right)\left(x,y\right)\right\}$$

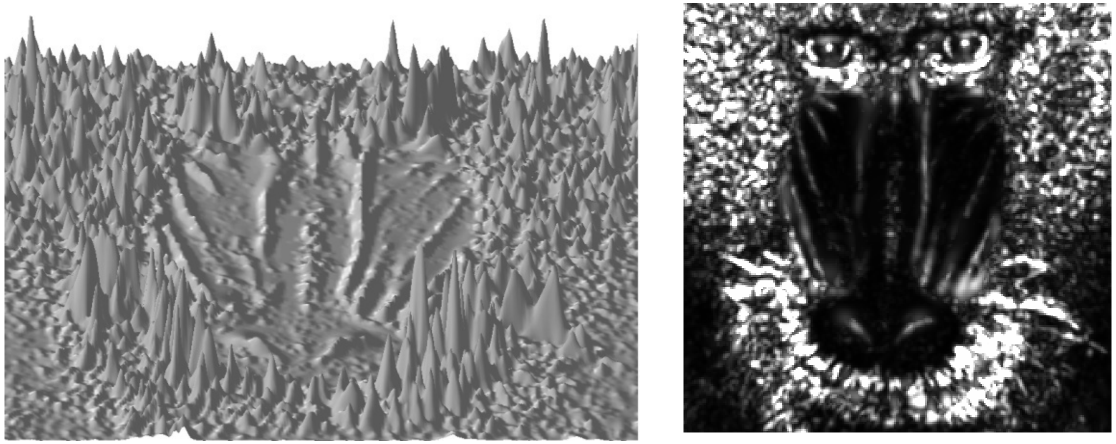
  - h_a: local averaging filter, h_k: the k-th Gabor filter-channel

- Indicates texture structure (analysis, detection, classification)
  - Criterion for the extraction of the texture dominant component
  - Dominant modulation features (amplitude, frequency, energy)

image



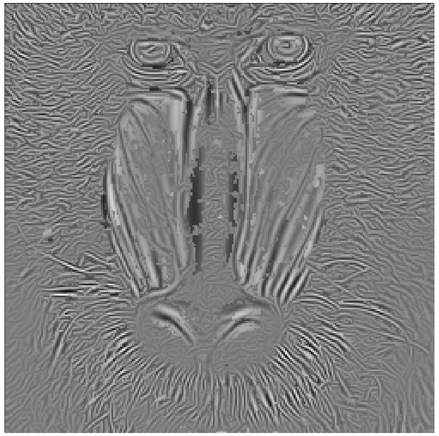
MAT energy



3D

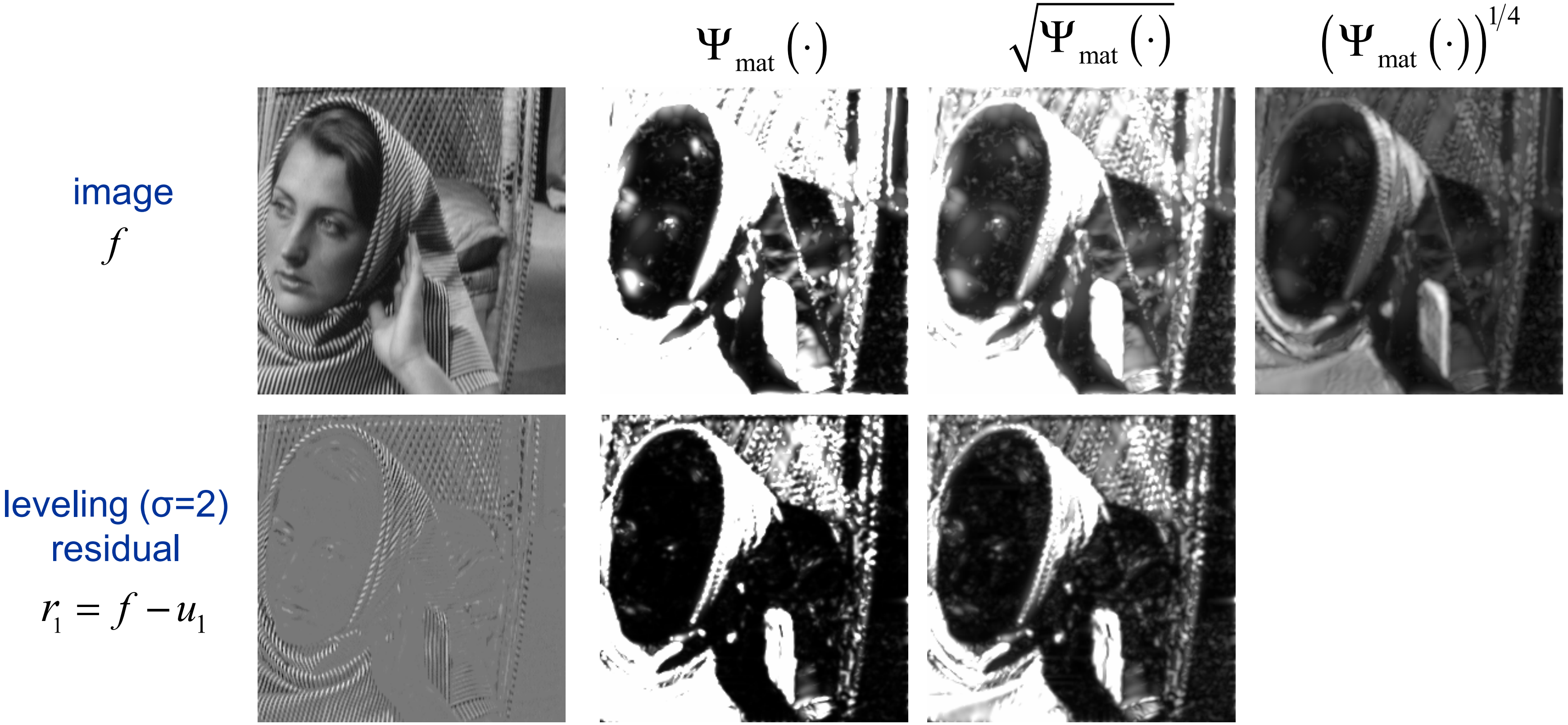
grayscale

dominant component



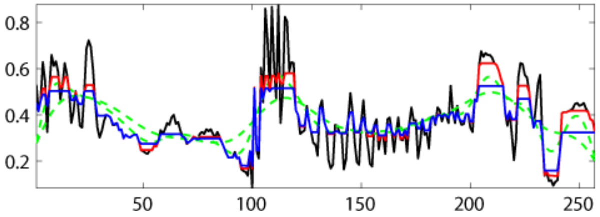
MAT energy for Texture detection

- Texture energy measurements for texture markers
  - indicate texture areas
  - quantify region ‘texturdeness’
  - roots of the MAT energy
- Markers extracted from the texture image part (e.g. leveling residual)
  - absence of large scale, geometric structures and features (edges, contours, blobs, contrast)
  - *f*=*u* + *v*: texture + objects, *v*: texture, details, oscillations



Leveling-based decomposition (Gaussian Markers)

- Cartoon = second-order leveling by Gaussian markers
  - ☐ *1st level* $u_1 = \Lambda\left(M_1 \mid f\right),\qquad M_1 = f * G_{\sigma 1}\qquad r_1 = f - u_1$
  - ☐ *2nd level* $\boxed{u = u_2 = \Lambda\left(M_2 \mid u_1\right),}\qquad M_2 = f * G_{\sigma 2}$
- Texture = residual from ‘leveling on residual’
  - ☐ *1st level* $u_r = \Lambda\left(M_3 \mid r_1\right),\qquad M_3 = r_1 * G_{\sigma 3},\quad \sigma 3 = \sigma 1/2$
  - ☐ *residual* $\boxed{v = r_1 - u_r}$



f

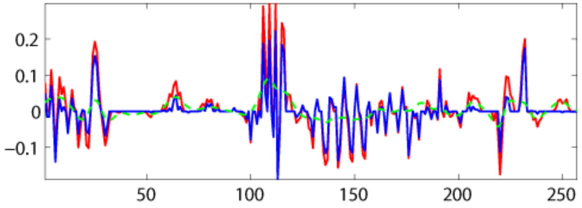
1st level

u_1

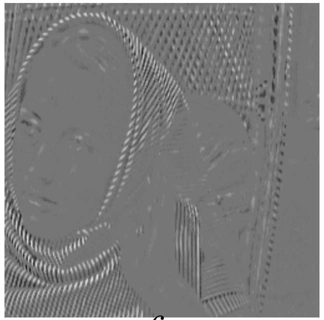
cartoon

u

$u_1 - u_2$



residual



$r_1 = f - u_1$

u_r

texture

v

reconstruction

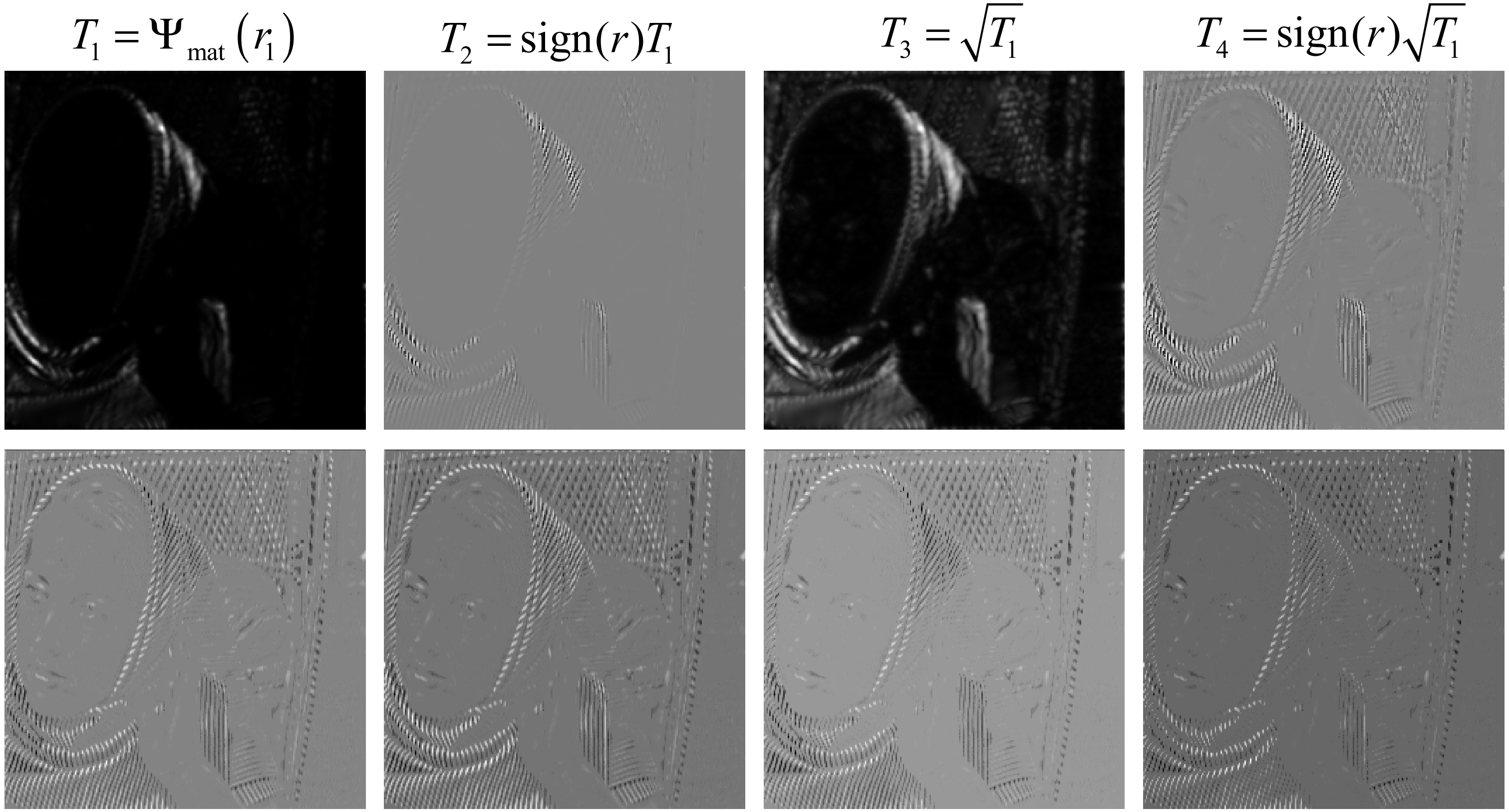
$u + v$

Leveling-based decomposition (Energy markers)

- Texture component v is retrieved by
  a. leveling the 'cartoon' residual $r_1 = f - \Lambda\left(M_1 \mid f\right)$ using texture-based markers
  b. keeping the 'new' residual

$$\boxed{v = r_1 - \Lambda\left(\operatorname{sign}(r_1)\left[\left(\Psi_{\text{mat}}\left(r_1\right)\right)^{1/k}\right] \mid r_1\right), \quad k = 1,2\ldots}$$

- Energy-based texture markers
  - ☐ mappings/transforms of the texture MAT operator (e.g. signed roots)


$T_1 = \Psi_{\text{mat}}\left(r_1\right)$

$T_2 = \operatorname{sign}(r)T_1$

$T_3 = \sqrt{T_1}$

$T_4 = \operatorname{sign}(r)\sqrt{T_1}$

Markers
T

Texture components
$v = r_1 \text{-} \Lambda\left(T \mid r_1\right)$

Application (Prehistoric Wall-Painting Restoration)

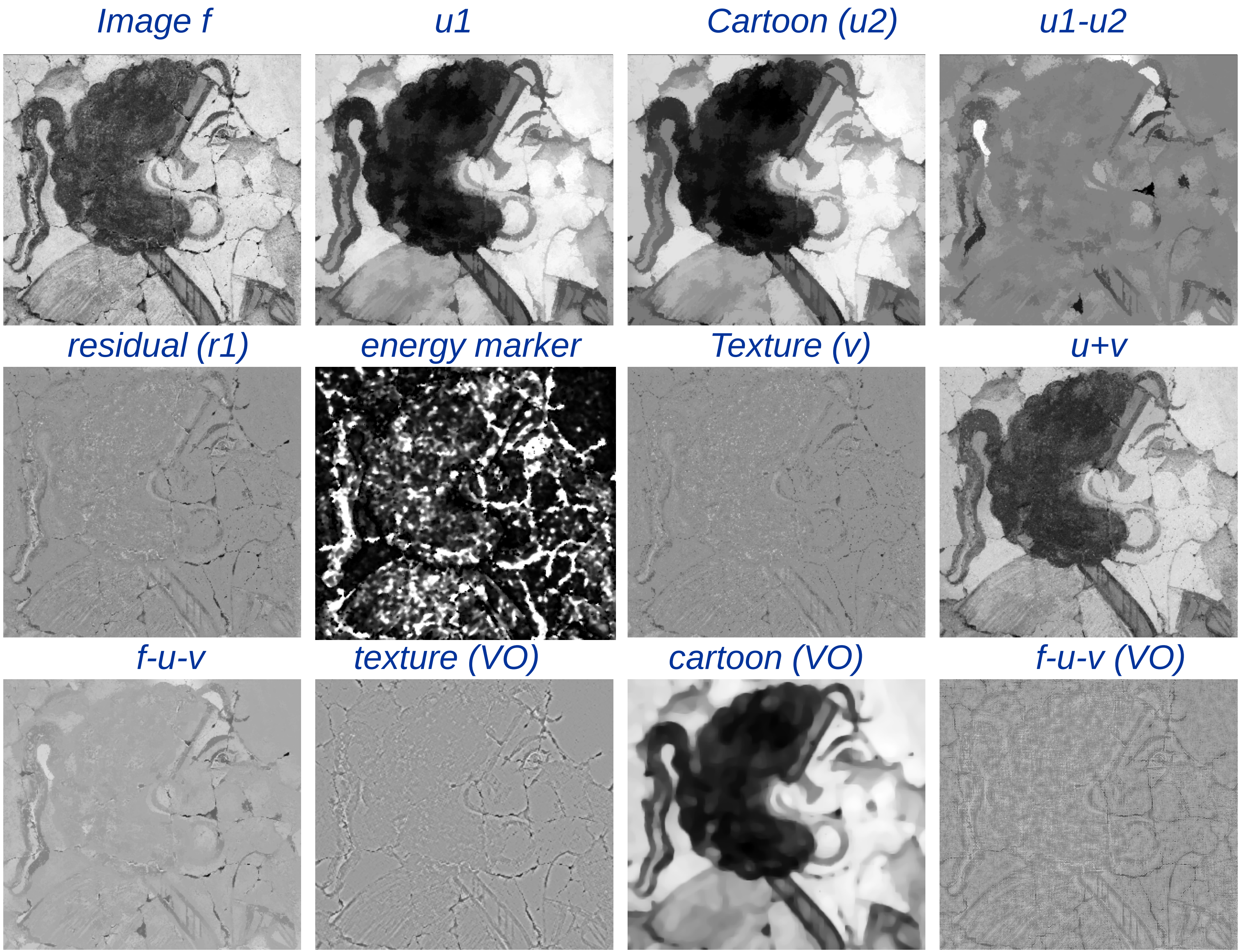


Image f *u1* *Cartoon (u2)* *u1-u2*

residual (r1) *energy marker* *Texture (v)* *u+v*

f-u-v *texture (VO)* *cartoon (VO)* *f-u-v (VO)*

Section of 'Potnia' prehistoric wall painting in Thira, Acrotiri

Generalized Watershed and PDEs for Geometric-Textural Segmentation

- ✓ Image Preprocessing and Simplification
- ✓ Image Decomposition into Constituent Components
- ✓ Feature Extraction
- ✓ Generalized Watershed and PDEs
- ✓ Coupled Contrast-Texture Segmentation

Sofou & Maragos, IEEE T-IP, 2008

IMAGE DECOMPOSITION INTO CONSTITUENT COMPONENTS

$f = u + v + w$

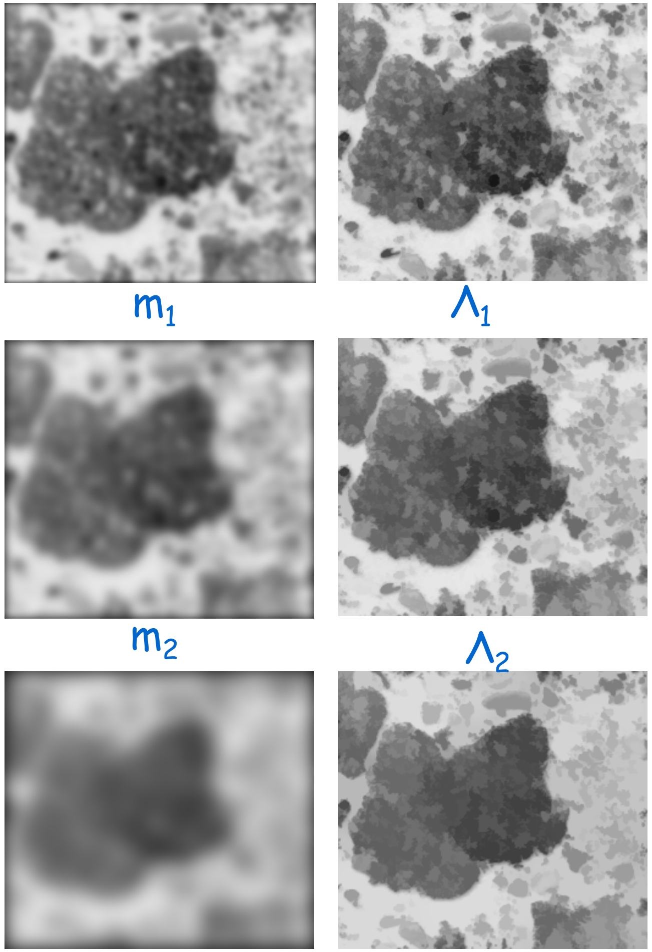
Image = geometrical structure + texture + noise

u : cartoon, v : texture, w : noise

$$u_1 = \Lambda(m_1 \mid f), \ldots, u_n = \Lambda(m_n \mid u_{n-1})$$

$$\boxed{u = \Lambda(m \mid f), \quad v = f - \Lambda}$$

Levelings Pyramid



Cartoon u: geometrical structure information, partly smooth with flat plateaus

Texture v: texture information, texture oscillations (quick variation of intensity)

u+v Decomposition

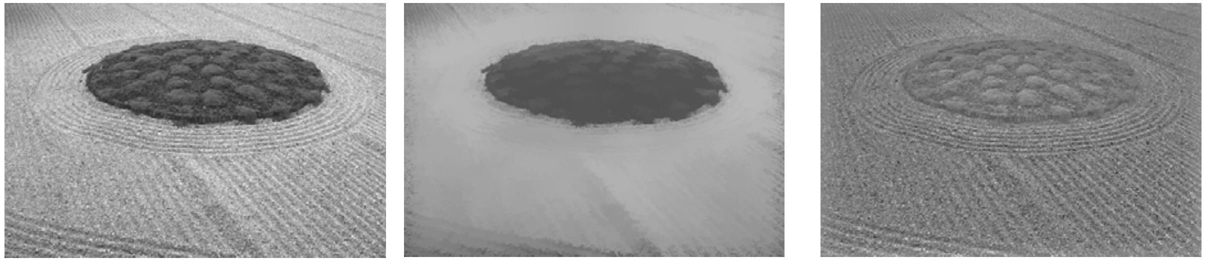
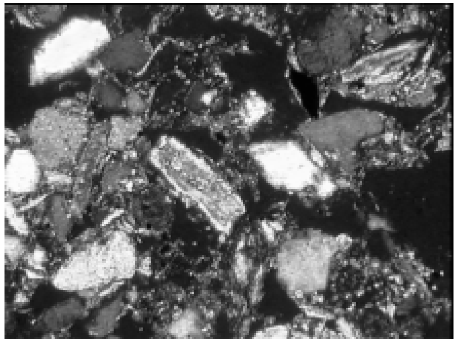
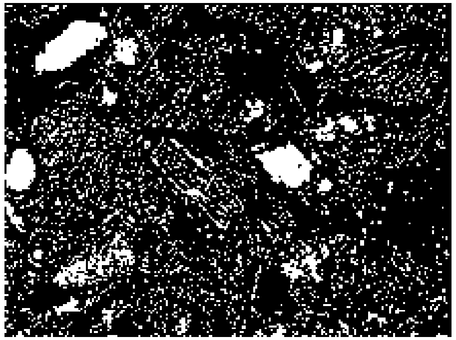
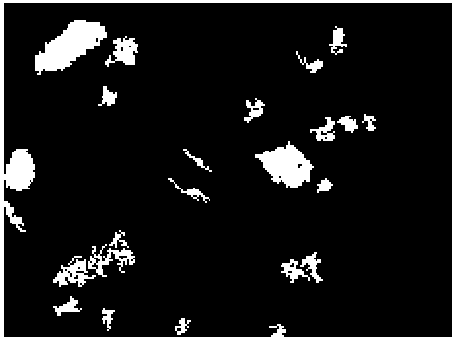
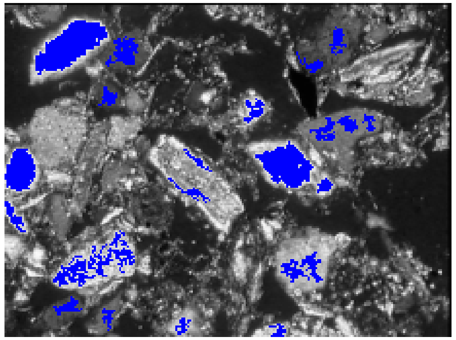
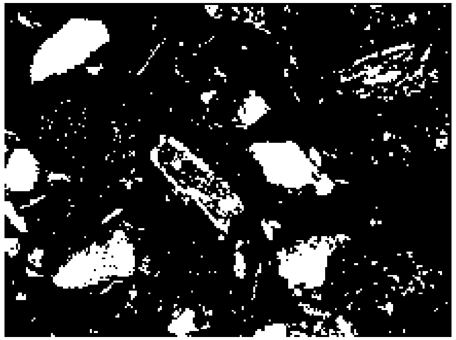
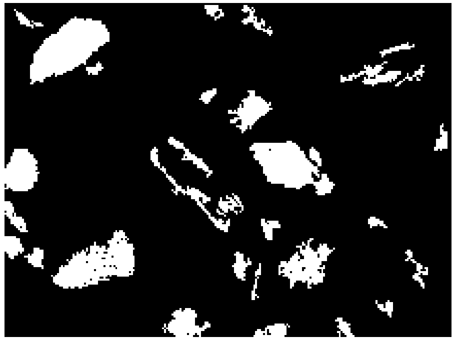
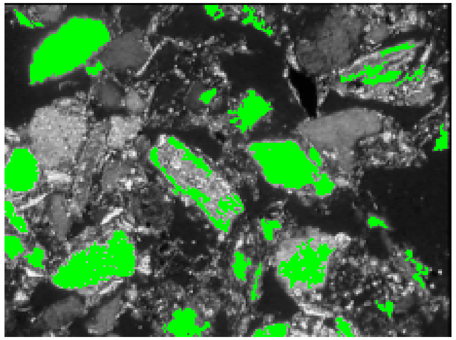
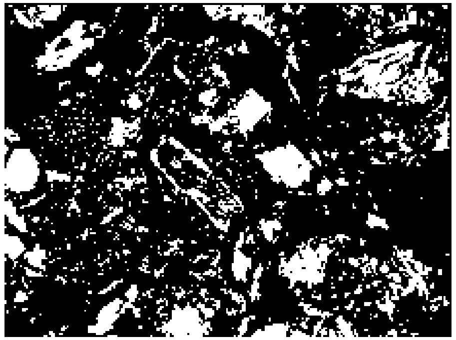
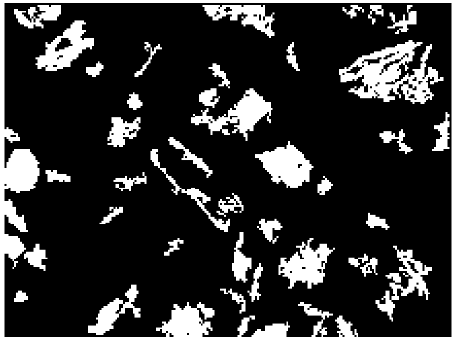
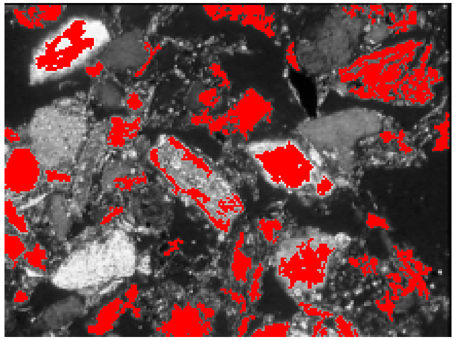


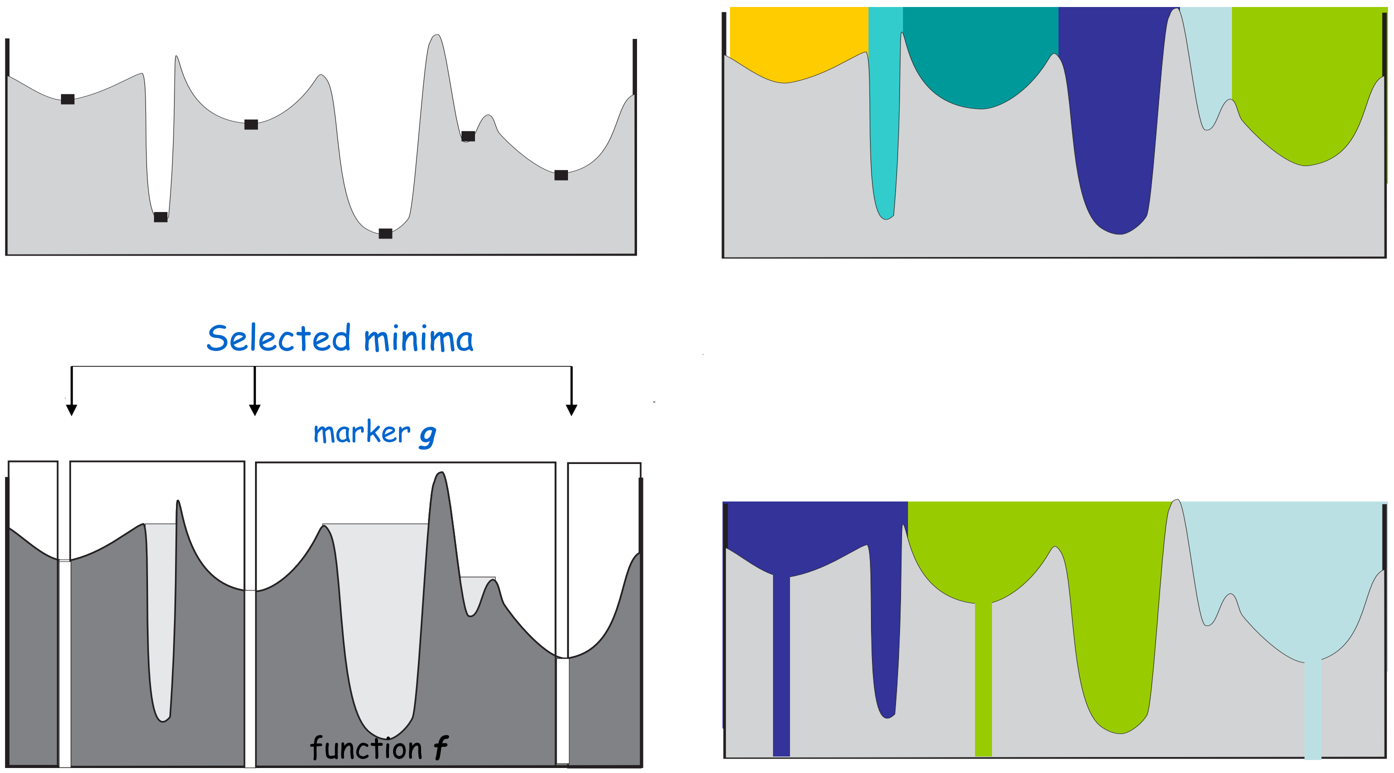
Image Leveling cartoon Texture

Image Leveling cartoon Texture

REGION MARKERS

Reference Image		Marker Set	Refined Marker Set	Marker Set Superimposed on Image
	Intensity Peaks			
	Region Peaks of certain Area			
	Region Peaks of certain Volume			

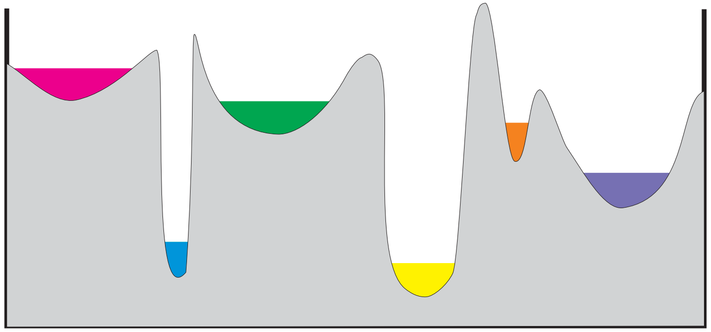
FLOODING PROCESS



- The gradient image is flooded from pre-selected sources (marker set).
- A lake is created from each flooding source.
- The water altitude rises inside each lake.
- The segmentation boundaries are formed at points where the emanating waves meet.

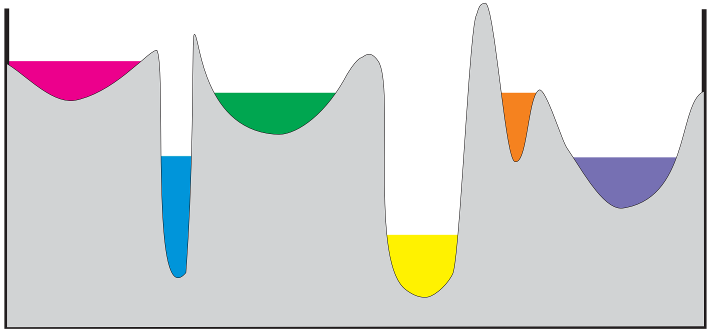
FLOODING CRITERIA AND TYPES OF WATERSHED FLOODING

Flooding Criterion: characteristic that all lakes (associated with the flooding sources) share with respect to water. By varying the flooding criterion different types of segmentation can be obtained.



Flooding with constant height criterion

- **Altitude /height** (contrast criteria)
  => Height Watershed Flooding.
- **Area** (size criteria)
  =>Area Watershed Flooding.
- **Volume** (contrast and area criteria)
  =>Volume Watershed Flooding.

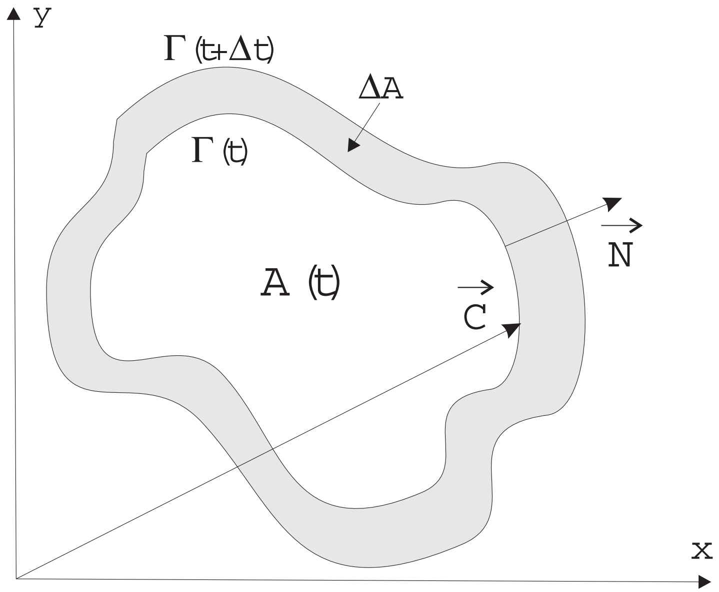


Flooding with constant volume criterion

UNIFORM HEIGHT FLOODING - 2D CASE

$\Gamma(t)$: closed planar curve of the lake boundary

$\vec{C}(t)$: position vector of the closed planar curve



Planar projection of a lake of a 2D function

Level Curve Evolution PDE:

$$\boxed{\frac{\partial \vec{C}}{\partial t} = \frac{c}{\|\nabla f\|} \cdot \vec{N}}$$

Level Set formulation

$$\Gamma(t) = \{(x,y) : \phi(x,y,t) = 0\}$$

$\phi(x,y,t)$: evolving space function

Level Function Evolution PDE:

$$\boxed{\frac{\partial \phi}{\partial t} = V(x,y)\|\nabla \phi\|}$$

$V(x,y)$: space-dependent speed function given by

$$V(x,y) = \frac{c}{\|\nabla f(x,y)\|}$$

UNIFORM VOLUME FLOODING - 2D CASE

$\vec{C}$: wave emanating from a lake flooded under the constraint of uniform volume speed.

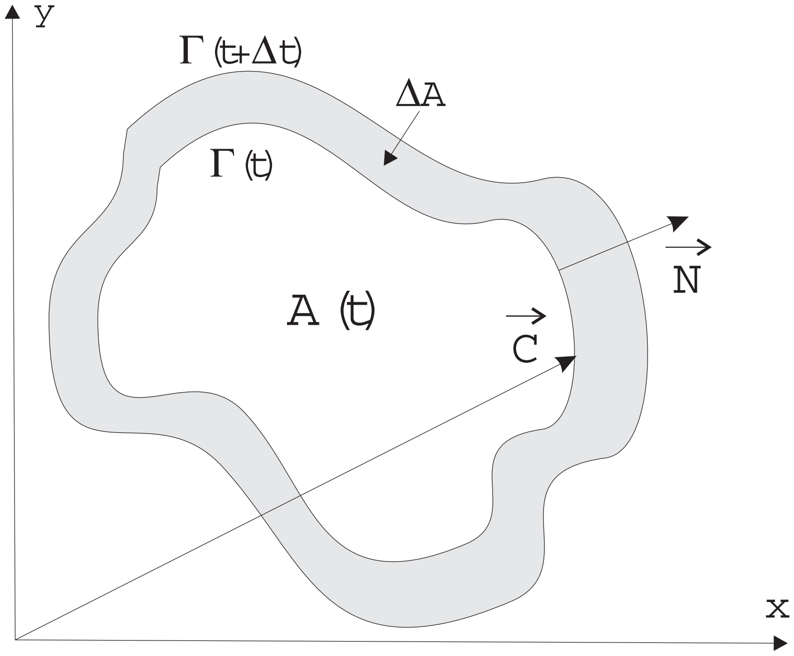
$L(t)$ becomes $A(t)$=> area enclosed by the propagating wave at time t

Level Curve Evolution PDE:

$$\boxed{\frac{\partial \vec{C}}{\partial t} = \frac{c}{A(t)\|\nabla f\|} \cdot \vec{N}}$$

Level Function Evolution PDE:

$$\boxed{\frac{\partial \phi}{\partial t} = V(x,y,t)\|\nabla \phi\|}$$

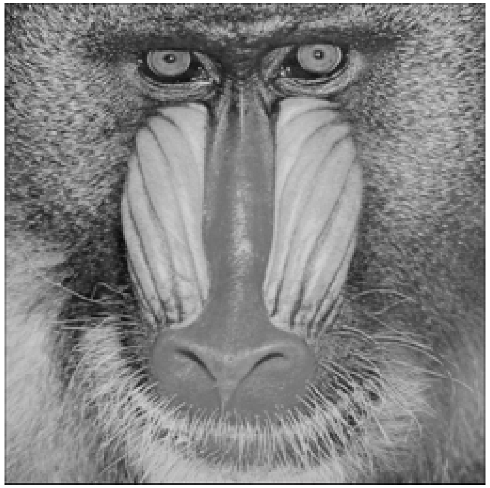


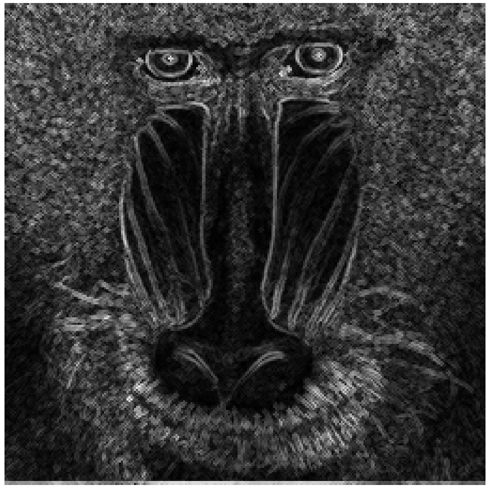
Planar projection of a lake of a 2D function

$$V(x,y,t) = \frac{c}{A(t)\|\nabla f(x,y)\|}$$

time and space dependent speed function

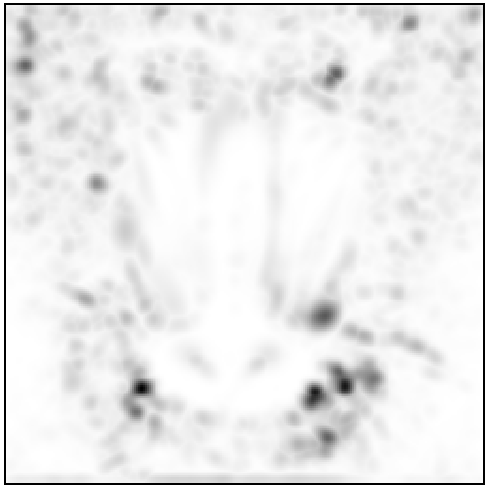
MULTI-CUE SEGMENTATION





Texture Quantification ?

Texture Modulation Energy



$$\frac{\partial \vec{C}}{\partial t} = \frac{c}{A(t)\left\|\nabla f\right\|}\cdot \vec{N} + \Psi_{\text{MAT}}(f)$$

texture

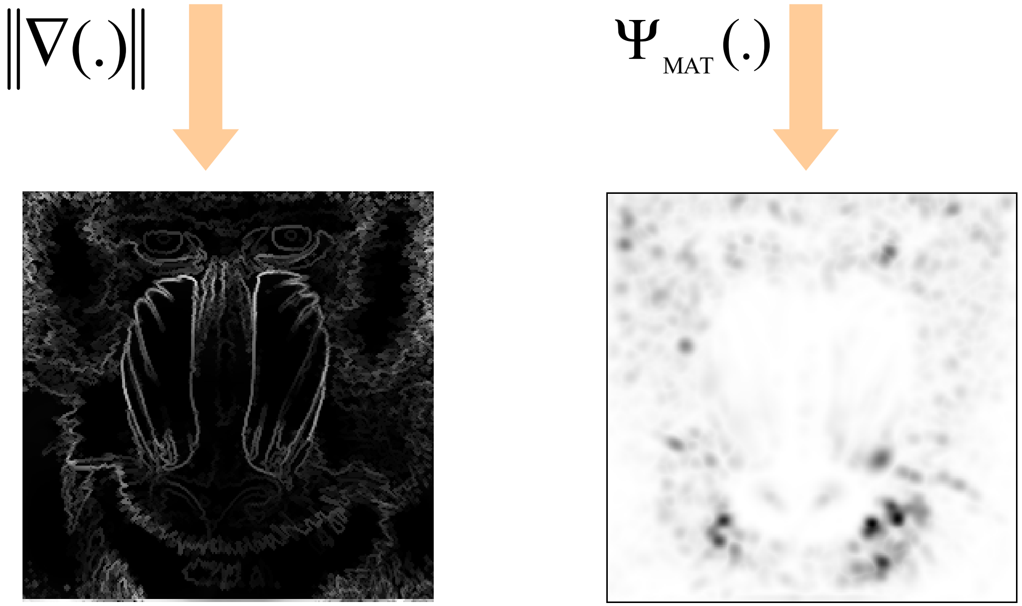
size

Intensity contrast

- Watershed flooding term (uniform height or volume) stops curve at strong edges
- Texture modulation energy term pushes curve away from areas of high energy without trapping it in-between texture edges

COUPLED MULTI-CUE SEGMENTATION

Component Decomposition $f = u + v$



Coupled Multicue segmentation Scheme

$$\frac{\partial \vec{C}}{\partial t} = \left(\frac{\lambda_1}{\mathrm{Area}(t)\|\nabla u\|} + \lambda_2 \Psi_{\mathrm{MAT}}(v) \right) \cdot \vec{N}$$

$$\frac{\partial \Phi}{\partial t} = \left(\frac{\lambda_1}{\mathrm{Area}(t)\|\nabla u\|} + \lambda_2 \Psi_{\mathrm{MAT}}(v) \right) \|\nabla \Phi\|$$

PARAMETER ESTIMATION

$$\frac{\partial \Phi}{\partial t} = \left(\frac{\lambda_1}{\text{Area}(t) \left\| \nabla u \right\|} + \lambda_2 \Psi_{\text{MAT}}(v) \right) \left\| \nabla \Phi \right\|$$

$$\boxed{\begin{aligned} \lambda_1(x,y) &= [G_\sigma * (f-v)^2](x,y) \\ \lambda_2(x,y) &= [G_\sigma * (f-u)^2](x,y) \end{aligned}}$$

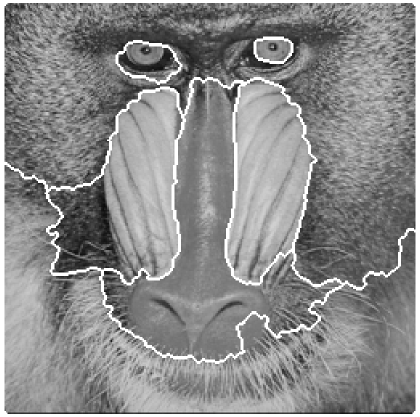
λ_1

λ_2

Normalization

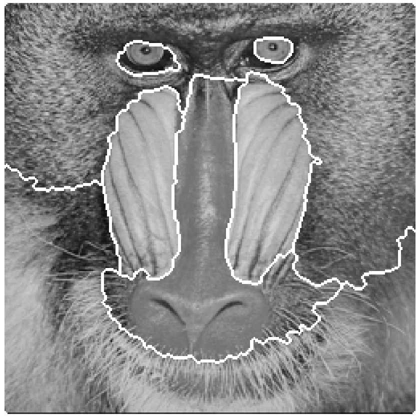
$$\boxed{\lambda_1 + \lambda_2 = 1}$$

$\lambda_1 = 0.3, \ \lambda_2 = 0.7$



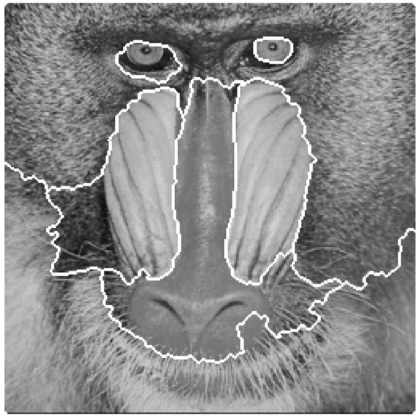
1.258 (opt fix)
Mumford- Shah quality criterion

$\lambda_1(x,y), \lambda_2(x,y)$



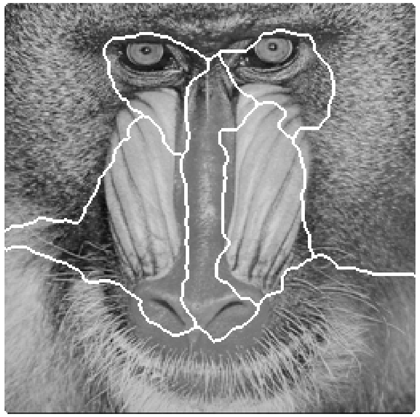
1.200
(adaptive)

$\lambda_1 = 1, \lambda_2 = 0$



1.259
(fixed)

$\lambda_1 = 0, \lambda_2 = 1$



1.315
(fixed)

QUALITY EVALUATION OF SEGMENTATION RESULTS

Liu –Yang Global Cost Function (LY)

$$F = \sqrt{N} \sum_{i=1}^{N} \frac{e_i^2}{\sqrt{\mathrm{Area_i}}}$$

N: number of regions,
$e_i^2 = (f - \mu_i)^2$

✓ tradeoff between preservation of level of detail and suppression of non-homogeneity.

✓ Punishes small regions, big number of regions and regions with high variance.

Mumford –Shah functional (MS)

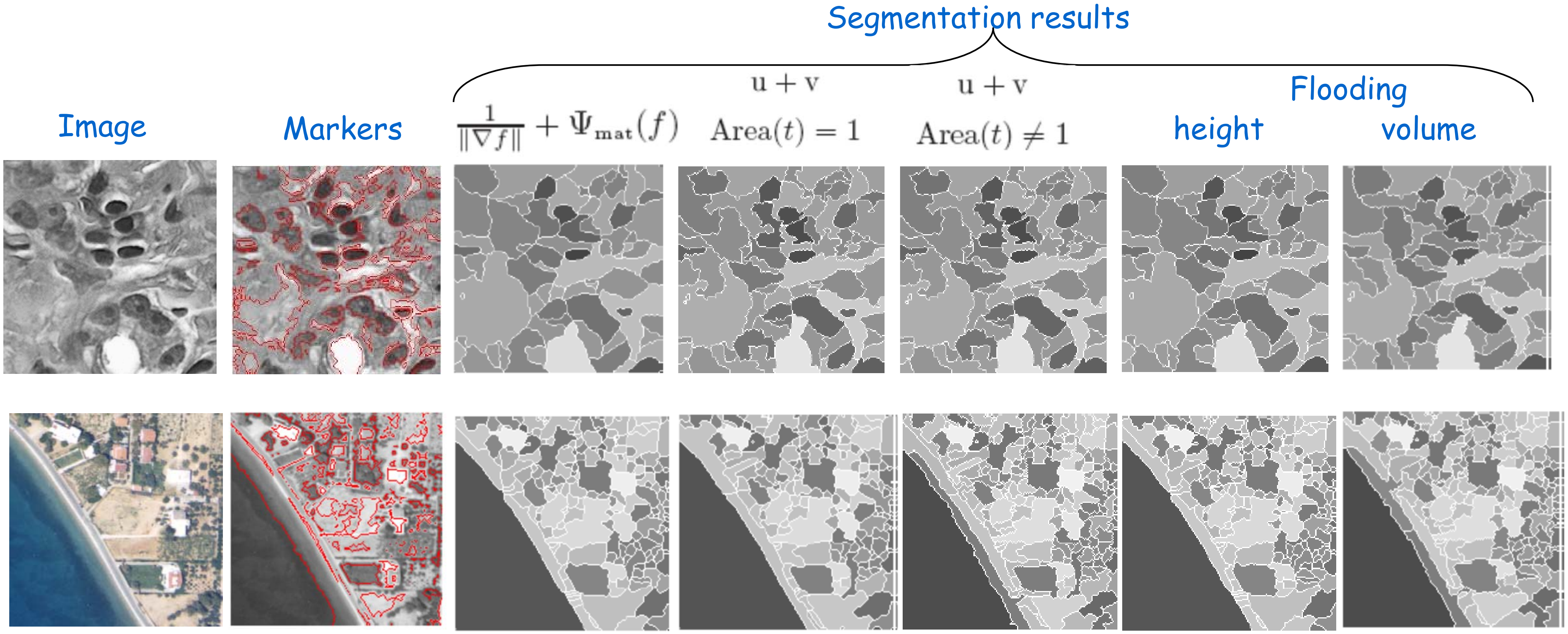
$$E(\Gamma, g) = \mu \iint_R (g - f)^2 dxdy + \xcancel{\iint_{R-\Gamma} \|\nabla g\|^2 dxdy} + \nu \, |\, \Gamma \,|$$

✓ region homogeneity
✓ smoothness of contours

g: smooth image → Mosaic Segmentation Image

Γ: region contours

SEGMENTATION RESULTS AND QUALITY MEASURES



		Segmentation Method				
		Multicue Segmentation			**Flooding**	
	Quality Criterion	$\frac{1}{\lVert\nabla f\rVert}+\Psi_{MAT}(f)$	$\frac{\mathrm{u+v}}{\mathrm{Area(t)=1}}$	$\frac{\mathrm{u+v}}{\mathrm{Area(t)\neq 1}}$	**Uniform height**	**Uniform volume**
Tissue image	**LY**	2.44	1.62	1.73	2.41	1.9
	MS	0.156	0.139	0.150	0.151	0.155
Aerial image	**LY**	2.9	2.42	1.11	2.95	1.21
	MS	0.182	0.170	0.182	0.184	0.185

REVISITING QUALITY CRITERIA

Selection of criteria that evaluate geometrical information as well as texture information

Total Weighted Variance of Cartoon component

$$\mathrm{var}(u) = \sum_{i=1}^{N} \sigma^2(u(R_i)) = \sum_{i=1}^{N} \frac{(u(x,y) - \overline{u}_i)^2}{|R_i|}$$

$$(x, y) \in R_i$$

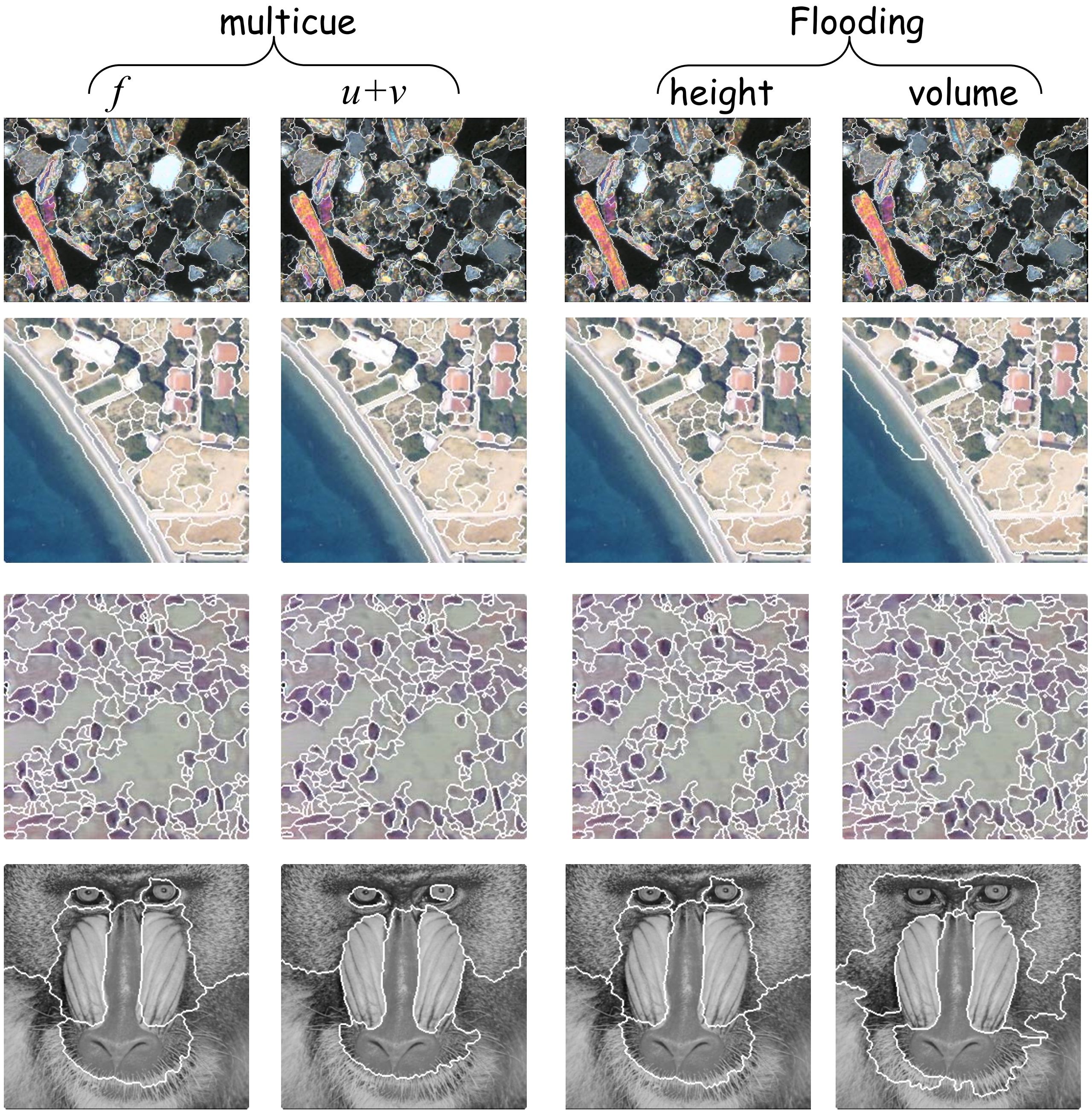
$|R_i|$ Region Cardinality

Total Weighted Variance of texture component

$$\mathrm{var}[\Psi_{\mathrm{MAT}}(v)] = \sum_{i=1}^{N} \sigma^2(\Psi_{\mathrm{MAT}}(v)(R_i)) = \sum_{i=1}^{N} \frac{([\Psi_{\mathrm{MAT}}(v)](x,y) - \mu_{\Psi i})^2}{|R_i|}$$

$\mu_{\Psi i}$ Mean texture modulation energy of the i-th region

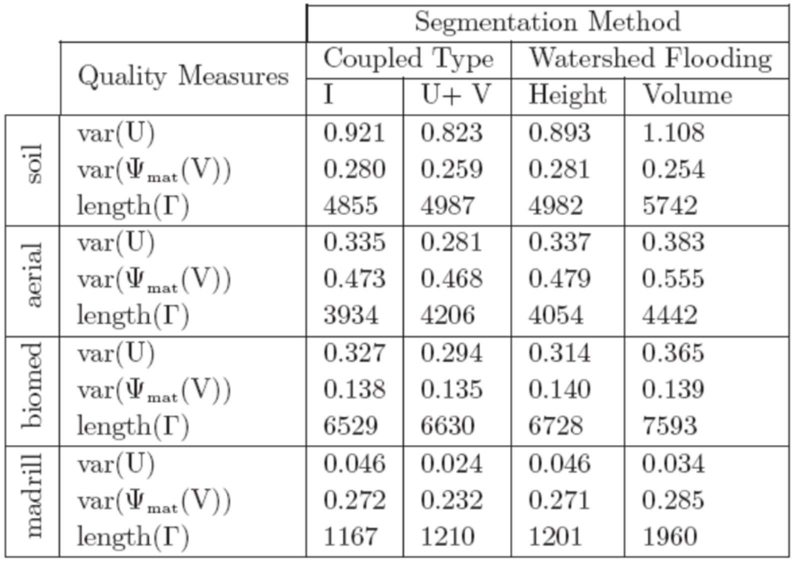
RESULTS



Comparison of region growing watershed-type methods

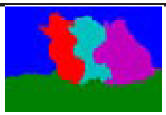
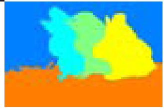
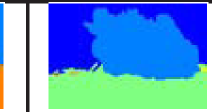
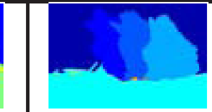
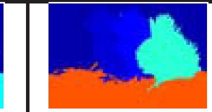
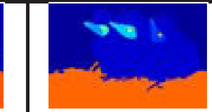
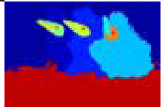
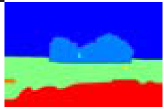
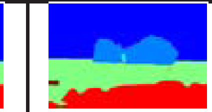
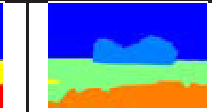
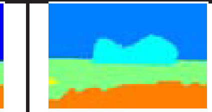
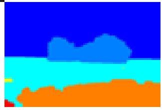
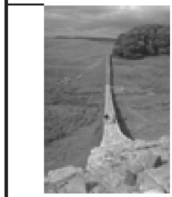
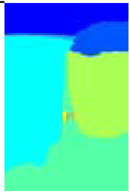
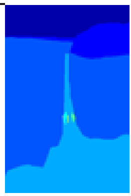
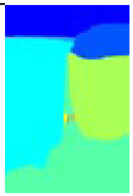
&

Quality criteria measurements



	Quality Measures	Segmentation Method			
		Coupled Type		Watershed Flooding	
		I	U+ V	Height	Volume
soil	var(U)	0.921	0.823	0.893	1.108
	var(Ψ_{mat}(V))	0.280	0.259	0.281	0.254
	length(Γ)	4855	4987	4982	5742
aerial	var(U)	0.335	0.281	0.337	0.383
	var(Ψ_{mat}(V))	0.473	0.468	0.479	0.555
	length(Γ)	3934	4206	4054	4442
biomed	var(U)	0.327	0.294	0.314	0.365
	var(Ψ_{mat}(V))	0.138	0.135	0.140	0.139
	length(Γ)	6529	6630	6728	7593
madrill	var(U)	0.046	0.024	0.046	0.034
	var(Ψ_{mat}(V))	0.272	0.232	0.271	0.285
	length(Γ)	1167	1210	1201	1960

COMPARISONS WITH GROUND TRUTH DATA

image	segmentation	Reference data (ground truth)						BCE
								0.21
								0.061
								0.16

Ground Truth data from Berkeley University Image Database

Texture Analysis & Segmentation Using Modulation Features, Generative Models, and Weighted Curve Evolution

Reference:

I. Kokkinos, G. Evangelopoulos & P. Maragos, IEEE T-PAMI 2009

Modulation-feature based Region Competition-GAR

- Efficient low-dimensional ***texture*** features for segmentation based on the **AM-FM** image model.

- ***Unsupervised segmentation*** using:
  - Curve Evolution & Level-Set methods combine efficiency & elegance
  - Region-based Terms guarantee robustness.

Advances in two directions

- ***Regularized demodulation***, for noise-discretization robustness.

- ***Probabilistic cue Integration*** based on generative models for features (edges vs. texture).

Kokkinos, Evangelopoulos & Maragos, IEEE T-PAMI, 2009

Modulation Features for Texture Analysis (I)

- Image AM-FM Modulation Model

$$I(x,y)=\sum_{k=1}^{K} a_k(x,y)\cdot\cos\left[\phi_k(x,y)\right], \qquad \nabla\phi_k(x,y)=\vec{\omega}_{\kappa}(x,y)$$

- Estimate instantaneous amplitude & frequency signals via
  - Multiband Gabor filtering $\longrightarrow$ Narrowband image components

$$I_k(x,y)=I(x,y)*h_k(x,y),$$

  - Demodulation using 2D Energy Operator $\Psi(I)\triangleq\|\nabla I\|^2-I\nabla^2 I$

and the ESA

$$\frac{\Psi\left(I_k\right)}{\sqrt{\Psi\left(\partial I_k/\partial x\right)+\Psi\left(\partial I_k/\partial y\right)}}\approx\left|a_k(x,y)\right|$$

$$\sqrt{\Psi\left(\partial I_k/\partial x\right)/\Psi(I_k)}\approx\left|\omega_{k1}(x,y)\right|, \qquad \sqrt{\Psi\left(\partial I_k/\partial y\right)/\Psi(I_k)}\approx\left|\omega_{k2}(x,y)\right|$$

Refs: Maragos & Bovik, *JOSA* 95, Daugman, *JOSA* 85, Havlicek, Harding & Bovik, 96, 00.

Modulation Features for Texture Analysis (II)

- *Dominant Components Analysis* (DCA) chooses at each pixel the most prominent channel, *j*

$$a(x,y) = \left|a_j(x,y)\right|, \qquad |\vec{\omega}(x,y)| = |\vec{\omega}_j(x,y)|$$

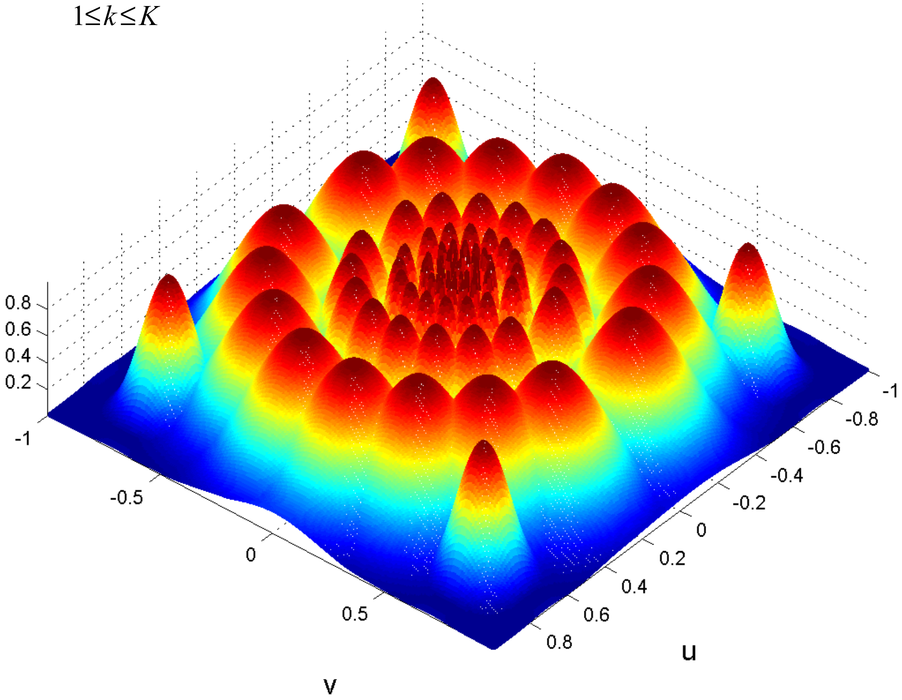
- Maximize criterion for choosing $j = \arg\max_{1\le k\le K}\{\Gamma_k\}$, among ***K*** channels

Amplitude-DCA

$$\Gamma_k(x,y) = \frac{\left|a_k(x,y)\right|}{\max_{\vec{\omega}} |H_k(\vec{\omega})|}$$

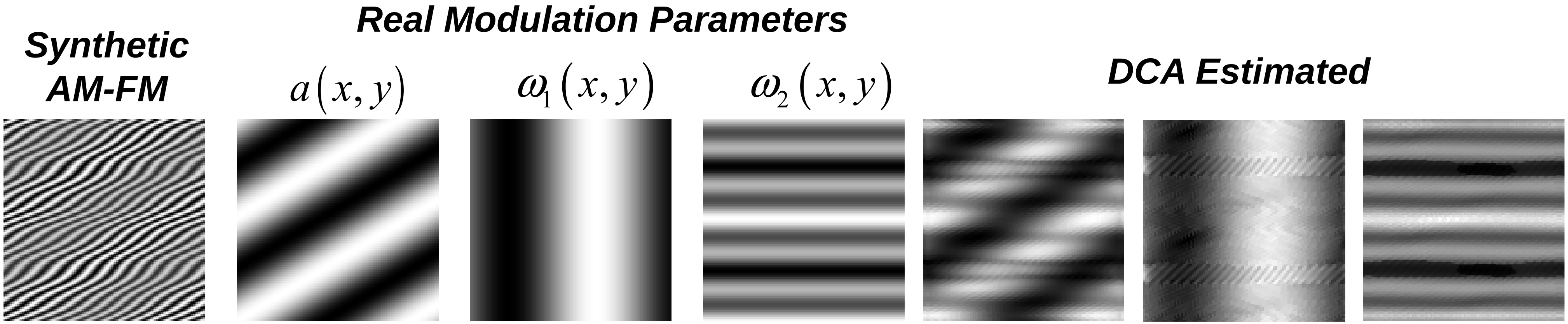
Teager *Energy*-DCA

$$\Gamma_k(x,y) = \Psi\left[\left(I * h_k\right)(x,y)\right]$$



- Using a single channel amounts to locally modeling the texture with a Gabor-like '*texton*' whose characteristics are described by the DCA components.

Modulation Feature Extraction Examples

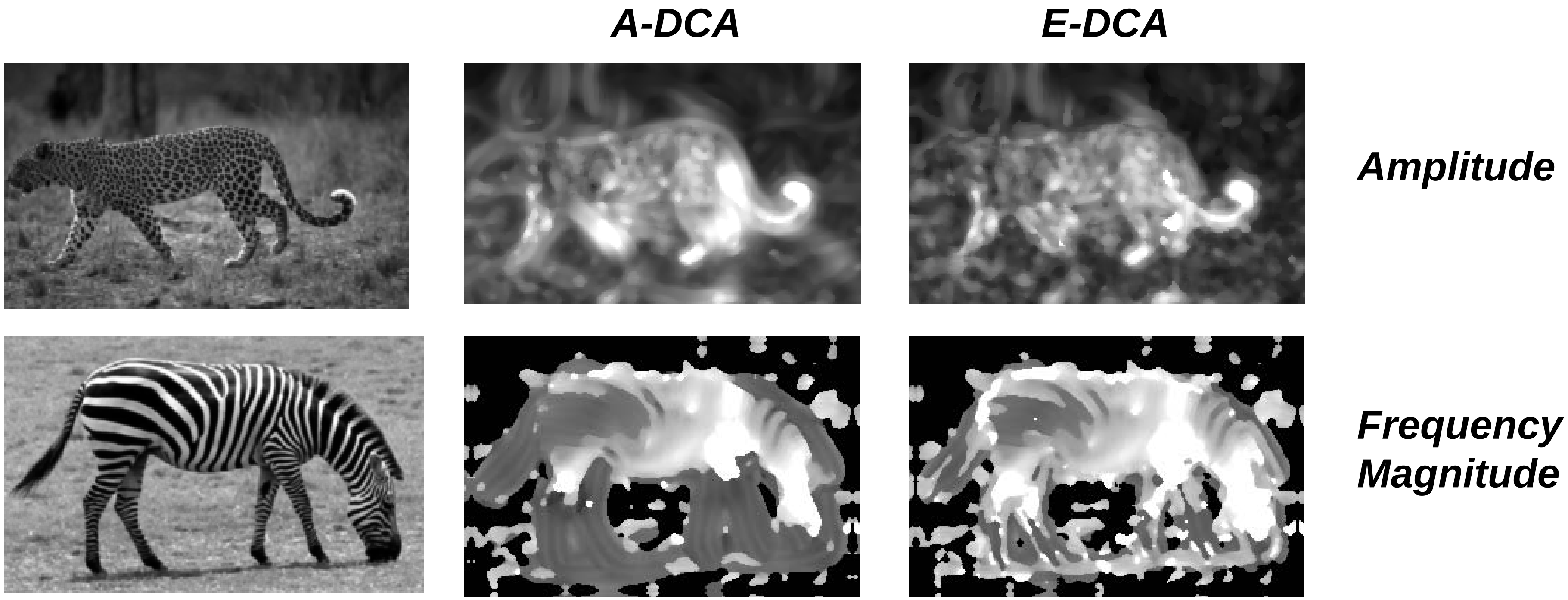


Synthetic AM-FM

Real Modulation Parameters

$a\left(x,y\right)$ $\omega_1\left(x,y\right)$ $\omega_2\left(x,y\right)$

DCA Estimated



A-DCA

E-DCA

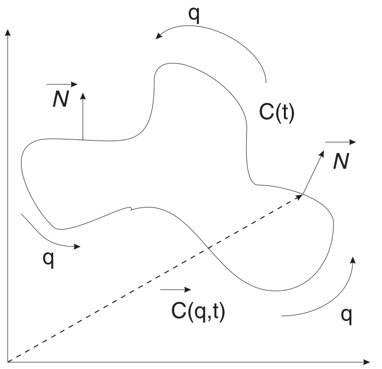
Amplitude

Frequency Magnitude

Unsupervised Variational Texture Segmentation

- Functional expressing segmentation cost (*Region Competition*):

$$J[C,\{\theta_i\}] = \sum_{i=1}^{M} \frac{\mu}{2} \int_{C_i} ds - \iint_{R_i} \log\left(P\left(I;\theta_i\right)\right) \qquad C = \{C_1,\dots,C_M\}$$

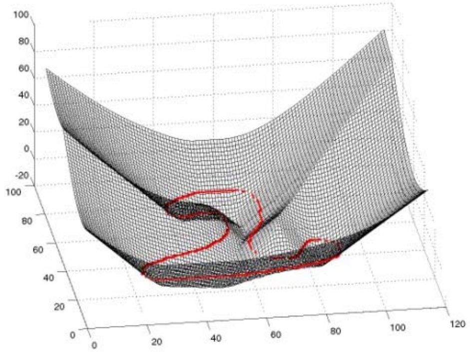


- *Euler-Lagrange* equations:

$$\frac{\partial C_i}{\partial t} = -\mu\kappa\vec{N} + \log\frac{P\left(I;\theta_i\right)}{P\left(I;\theta_j\right)}\vec{N}$$

- *Level Set* Implementation & Edge-based terms (*Geodesic Active Regions*):

$$\frac{\partial C_i}{\partial t} = \lambda \log\frac{P\left(I;\theta_i\right)}{P\left(I;\theta_j\right)}\vec{N} - (1-\lambda)\left[g(\mathrm{I})k\vec{N} + \left(\nabla g(\mathrm{I})\cdot\vec{N}\right)\vec{N}\right]$$



- *Active Contours without Edges, Statistical approach to Snakes*

Refs: Zhu & Yuille, T-*PAMI* 96, Paragios & Deriche *IJCV* 02, Vese & Chan T-*IP* 01, Yezzi, Chai & Willsky *ICCV* 99

DCA Features for Unsupervised Variational Segmentation

- DCA Features provide a low-dimensional and rich texture descriptor, that contains local information about
  - *Oscillation Amplitude,* *(Contrast)*
  - *Frequency* Magnitude (Scale)
  - Direction of Phase Variation (*Orientation)*

- Features Used for Segmentation: $[a, \omega_1, \omega_2, I]^{\mathrm{T}}$ **(OR)** $[a, |\vec{\omega}|, \angle\vec{\omega}, I]^{\mathrm{T}}$

- Distribution $P(.;\theta_i)$ in each region is modeled as a
  - ☐ multivariate Gaussian for $[I, a, |\vec{\omega}|]$
  - ☐ von-Mises for the orientation $\angle\vec{\omega}$

- Initialize and Iterate:
  - ☐ Estimate the parameters for each region, using the front's current position
  - ☐ Evolve fronts in the direction dictated by region competition

    (statistics force + geometrical information)

2D Gabor ESA

- 2D energy operator with Gabor bandpass filtering

$$f(x,y) = I(x,y) * h(x,y)$$

- Gabor Energy Operator

$$\Psi(f) = \Psi(I*h) = \left\| I*\nabla h \right\|^2 - (I*h)\left(I*\nabla^2 h\right)$$

- Differential operators are replaced by derivatives of Gabor

- Estimation of inst. amplitude and frequency by ESA

$$\Psi(f_x = I*h_x) = \left\| I*\nabla h_x \right\|^2 - (I*h_x)\left(I*\nabla^2 h_x\right), \quad \Psi\left(f_y = I*h_y\right)$$

- 2D Gabor ESA: need seven Gabor differential formulae

$$\left(h_x, h_y, h_{xx}, h_{yy}, h_{xy}, \nabla^2 h_x, \nabla^2 h_y\right)$$

Regularized ESA

- Reduce complexity of applying Gabor ESA to all filters
- Bandpass Image $f_k(x,y) = I(x,y) * h_k(x,y)$
- Regularized Energy Operator

$$\Psi_\sigma(f_k) = \left\| f_k * \nabla G_\sigma \right\|^2 - f_k \left(f_k * \nabla^2 G_\sigma \right)$$

REO needs three convolutions of f_k with $\partial / \partial x, \partial / \partial y, \nabla^2$ of the Gaussian

- Apply Regularized ESA to each channel

$$\Psi_\sigma(\partial f_k / \partial x) = \left\| f_k * \nabla(\partial_x G_\sigma) \right\|^2 - f_k \left[f_k * \nabla^2 (\partial_x G_\sigma) \right], \qquad \Psi_\sigma(\partial f_k / \partial y)$$

Cue Integration Problem

- Ubiquitous zebra image:

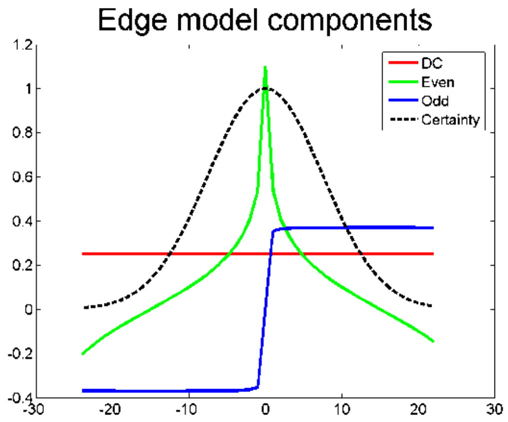
  - In the interior, ∇g can get the curve evolution stuck on one of the stripes.
  - Along the object border the estimated amplitude of A is large and impedes the accurate localization of the borders.

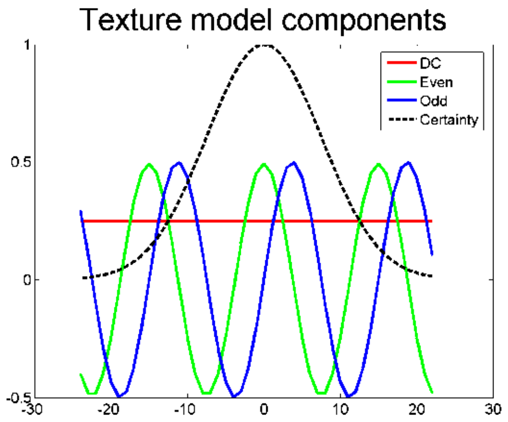
- How can we combine different cues in an automated manner?

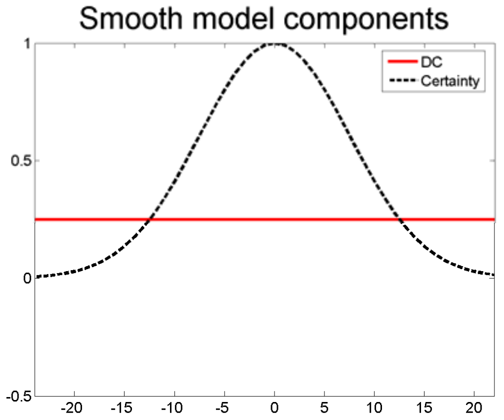
- The usefulness of features depends on the validity of the underlying model (if any).

Probability Assignments to Features

- Three simple classes of generative models corresponding to the texture/edge/smooth local image models are used


Edge model components


Texture model components


Smooth model components

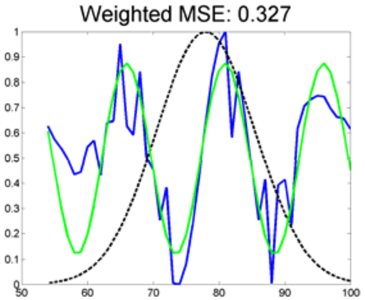
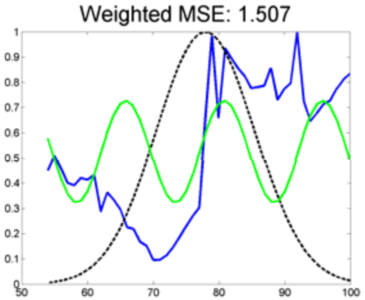
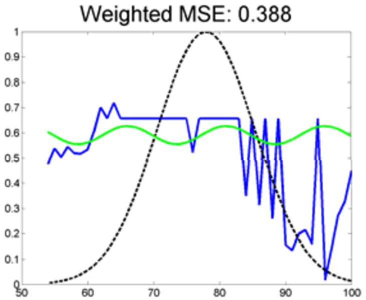
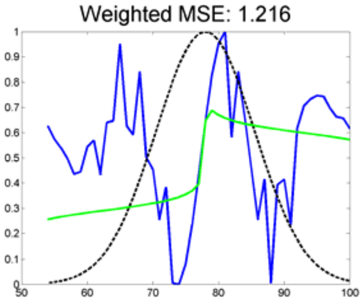
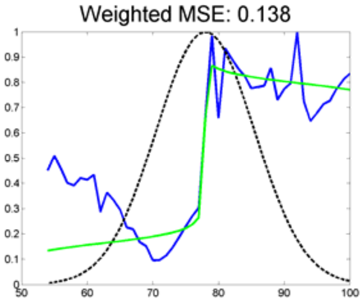
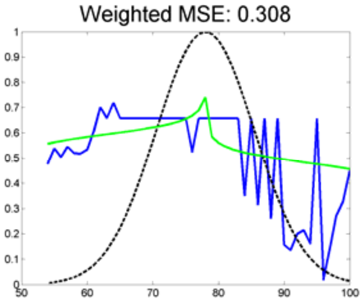
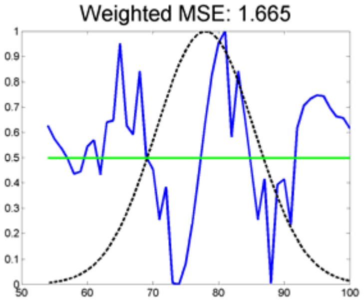
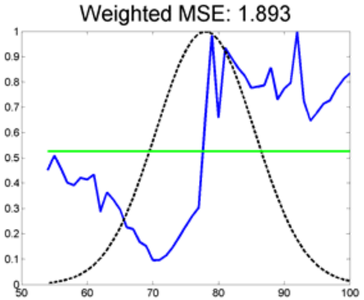
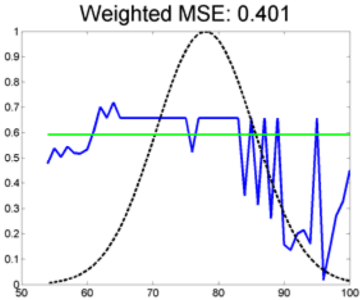
- A spatially varying confidence measure accounts for locality.

$$P(I\,|\,T,0)=\prod_{x}\left[G(x)P(I(x)\,|\,S^{T}(x))+(1-G(x))c\right]$$

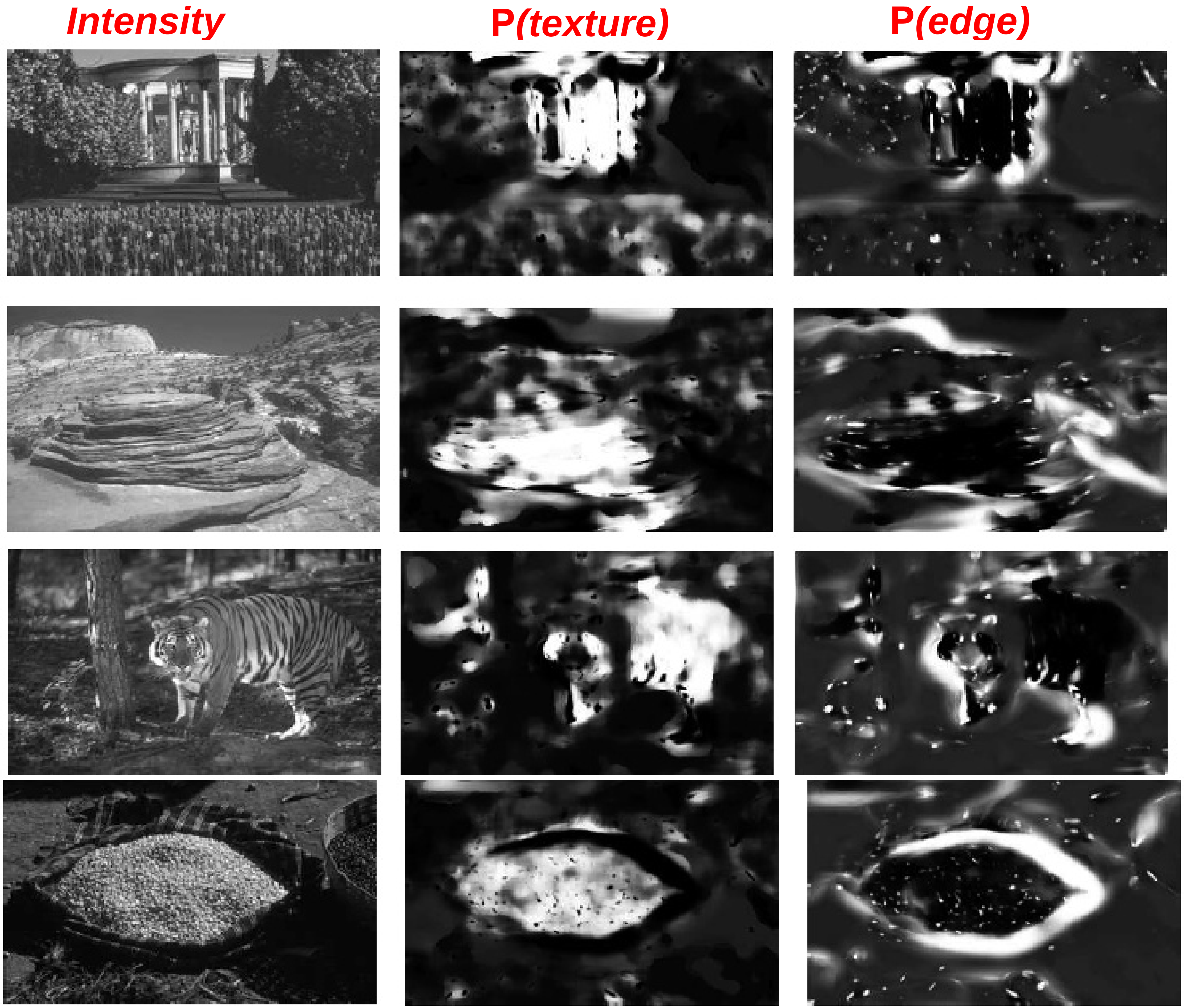
- A lower bound on the log likelihood can be estimated using convolution operations
- A ‘posterior probability’ for each model class is derived, offering a confidence measure for the related features.
- Simple and efficient method for deriving confidence measures.

Refs: Kokkinos, Evangelopoulos, Maragos, ICIP 04 Kokkinos & Maragos, Texture workshop, 05

Model-based Signal/Hypothesis Reconstruction

Signal / Model	Texture	Edge	Smooth
Texture	 Weighted MSE: 0.327	 Weighted MSE: 1.507	 Weighted MSE: 0.388
Edge	 Weighted MSE: 1.216	 Weighted MSE: 0.138	 Weighted MSE: 0.308
Smooth	 Weighted MSE: 1.665	 Weighted MSE: 1.893	 Weighted MSE: 0.401

Model-based Cue Probabilities



Cue Integration for Region Competition

- How can we introduce the confidence measures in the evolution?

- Modified RC with probability assignments to features

w_c : cue weight, w_e : edge.

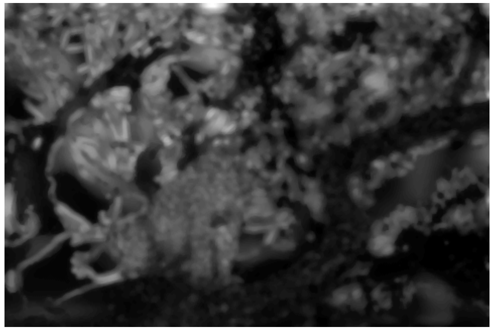
$$\frac{\partial C_i}{\partial t} = \sum_{c \in T,S} w_c \log \frac{P^c\left(F_c;\theta_i\right)}{P^c\left(F_c;\theta_j\right)} \vec{N} - w_e\left(\nabla g \cdot \vec{N}\right)\vec{N} - gk\vec{N}$$

Features and Segmentation

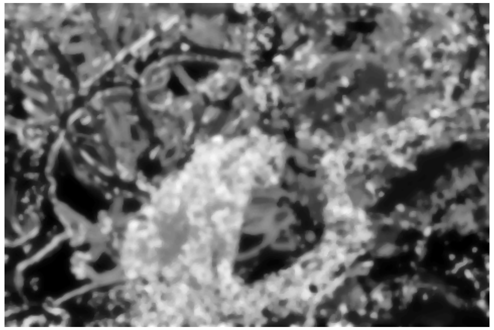
Features for Segmentation: Intensity, Amplitude, Freq. Magnitude, Freq. Orientation

I

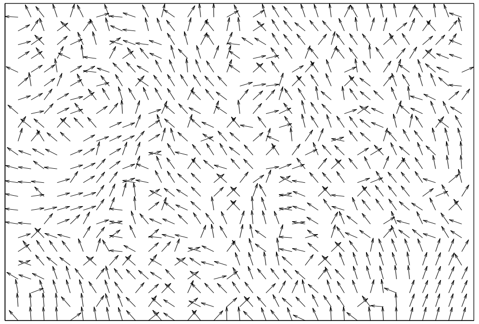
a



$|\vec{\omega}|$

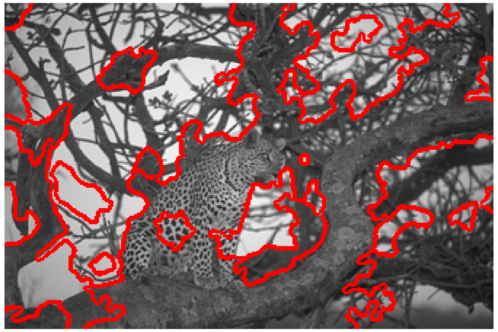


$\angle\vec{\omega}$



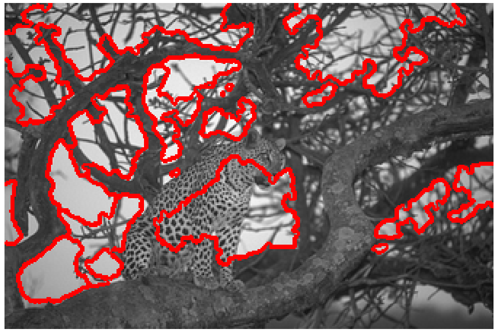
Segmentation Results - Comparisons

RC-GAR



Feats: $[a, |\vec{\omega}|, \angle\vec{\omega}, I]^{\mathrm{T}}$

Weighted RC-GAR

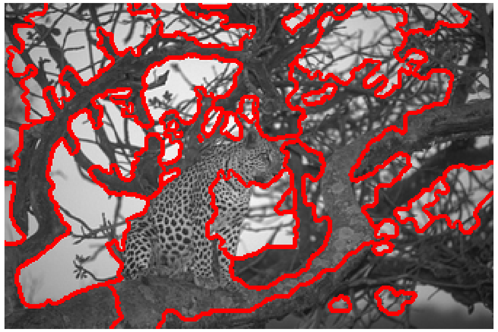


$[a, |\vec{\omega}|, I]^{\mathrm{T}}$

Weighted RC-GAR

$[a, |\vec{\omega}|, \angle\vec{\omega}, I]^{\mathrm{T}}$

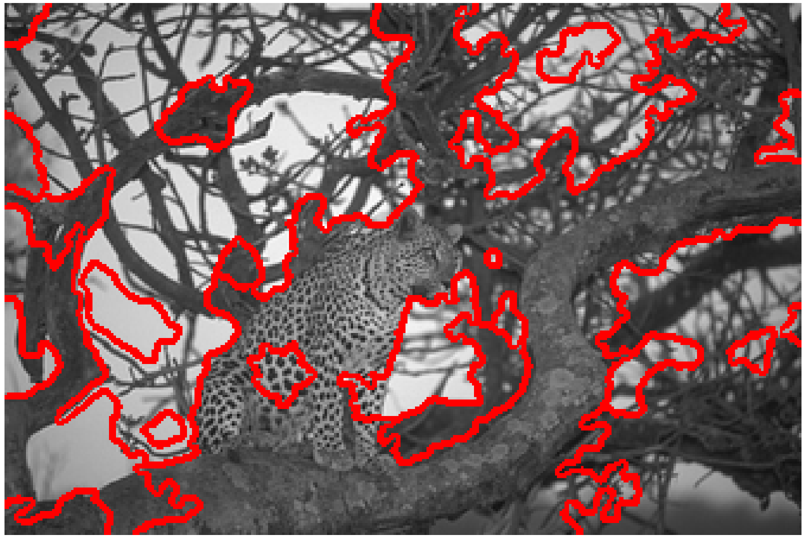
RC-GAR



Diffusion features

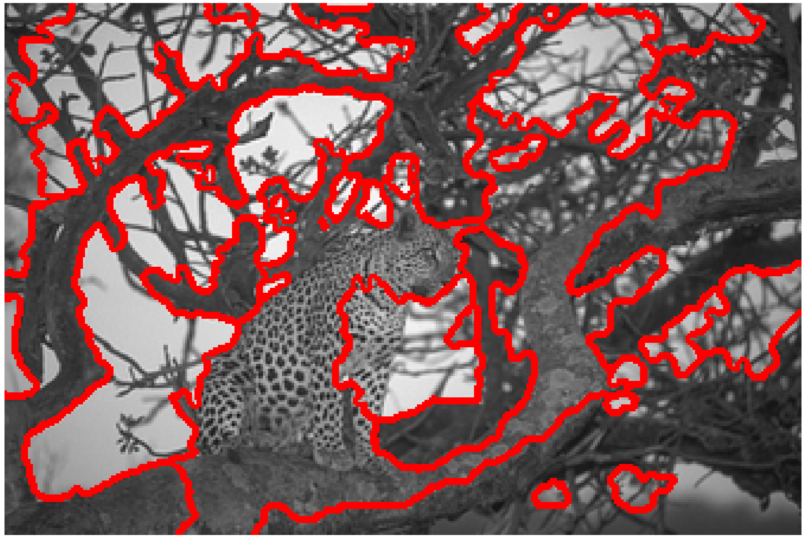
Segmentation Comparisons

RC-GAR+ $[a, |\vec{\omega}|, \angle\vec{\omega}, I]^{\mathrm{T}}$

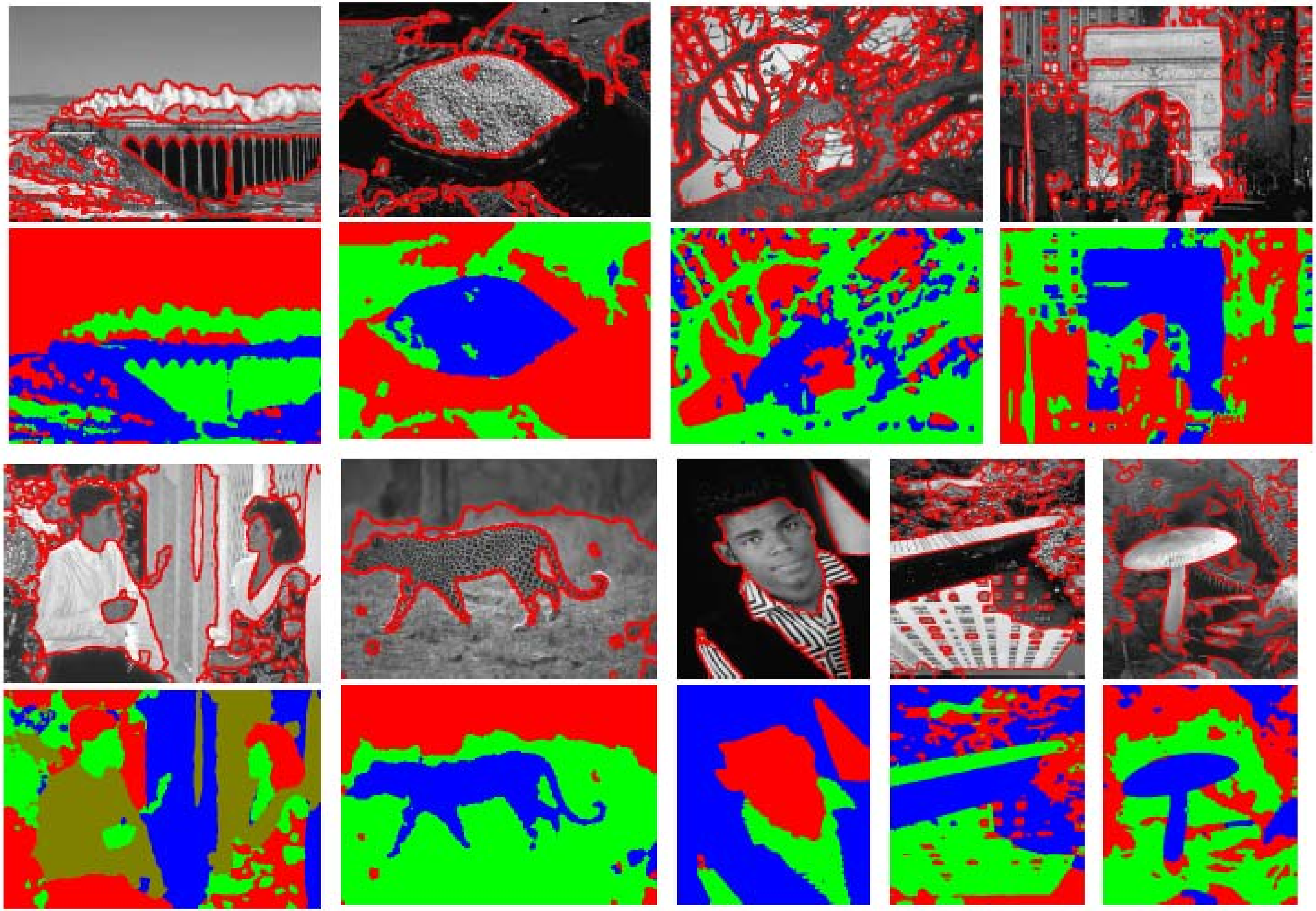


Weighted RC-GAR+ $[a, |\vec{\omega}|, \angle\vec{\omega}, I]^{\mathrm{T}}$

***RC-GAR+* Diffusion features**



Unsupervis. Segmentation w. Weighted Curve Evolution



Summary

- Texture representation by simple, information rich, low-dimensional feature vector.
- Unsupervised segmentation scheme that combines the merits of Region Competition/GAR and Modulation/DCA.
- Regularized modulation feature estimation (Gabor EO, REO).
- Probabilistic cue integration based on generative models for features (texture, edge, smooth areas) for improved AM-FM Region Competition (e.g. suppress orientation feature in smooth areas or amplitude along edges).
- Segmentation of wide variety of natural textures, systematic evaluation on the Berkeley test set for unsupervised segmentation.

http://cvsp.cs.ntua.gr

CONCLUSIONS

- Two Nonlinear Geometric Approaches for Image Analysis and Vision: Mathematical Morphology and Geometric PDEs/Variational .

- Applications to Multiscale Analysis and Segmentation Problems

- Morphological Operators: (i) Algebraic aspects and (ii) Variational Formulation and related PDEs.

- Links/Synergy between MM and nonlinear PDEs for vision .

- Multi-cue approaches

Future directions:

- Minimax Algebra: Weighted Lattices
- Graph Morphology: PdEs, ACs on Graphs
- Patch-based PDEs and variational problems

Multiscale Morphology and Geodesic Active Contours on Arbitrary Graphs

References:

- P. Maragos & K. Drakopoulos, Proc. 2011 Dagstuhl Symp. on Shape
- K. Drakopoulos & P. Maragos, IEEE JSTSP, 2012

Tensor-based Image Diffusions Derived from Generalizations of the Total Variation and Beltrami Functionals

29 September 2010

International Conference on Image Processing 2010, Hong Kong

Anastasios Roussos and Petros Maragos

Computer Vision, Speech Communication and Signal Processing Group,
School of Electrical & Computer Engineering,
National Technical University of Athens, Greece

http://cvsp.cs.ntua.gr

Motivation (1/2)

- Nonlinear diffusion models for Computer Vision
  - Class A: Directly-designed PDEs
    - Perona-Malik method [ieeeT-PAMI'90]
    - CLMC regularized PDE [Catte et al, siamJNA'92]
    - Coherence-enhancing diffusion [Weickert, IJCV'99]
    - Method of [Tschumperlé & Deriche, ieeeT-PAMI'05]

      ⋮
  - Class B: Variational Methods
    - Total Variation [Rudin, Osher & Fatemi, PhysicaD'92]
    - Vectorial Total Variation [Sapiro, CVIU'97]
    - Color Total Variation [Blomgren & Chan, ieeeT-IP'98]
    - Beltrami Flow [Sochen, Kimmel & Maladi, ieeeT-IP'98]

      ⋮
- For some methods of Class A: known connection with Class B, e.g. :
  - Perona-Malik model $\dfrac{\partial u(x,y,t)}{\partial t} = \operatorname{div}\left(g(\|\nabla u\|^2)\nabla u\right)$
  - $\min\limits_{u} \displaystyle\int_{\Omega} \varphi(\|\nabla u\|^2)\mathrm{d}\boldsymbol{x} \rightsquigarrow \dfrac{\partial u}{\partial t} = \operatorname{div}\left(2\varphi'(\|\nabla u\|^2)\nabla u\right)$

$$g(s^2) = 2\varphi'(s^2)$$

- But, for several types of PDE-based diffusion methods no variational interpretation existed

Motivation (2/2)

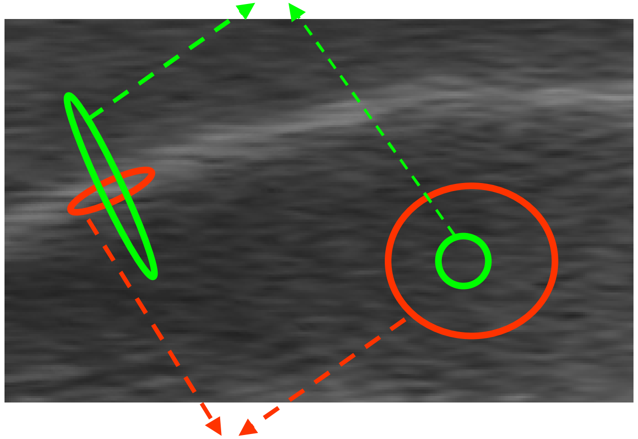
- Advantages of variational interpretation of diffusion methods
  - conceptually clear formalism
  - helps with the reduction of model parameters
  - easier application to problems that can be formulated as constrained energy minimization, e.g.:
    - image restoration, inpainting, interpolation
  - can lead to efficient implementations based on optimization techniques

- Advantages of using tensors in image diffusion
  - Structure tensor

    measure of the image variation & geometry in the neighborhood of each point
  - Diffusion tensor

    flexible adaptation to the image structures

structure tensor



diffusion tensor

Contributions

- We propose a novel generic functional that:
  - is designed for vector-valued images
  - generalizes several existing variational methods
  - is based on the structure tensor
  - leads to tensor-based nonlinear diffusions that contain regularizing convolutions

- As special cases, we propose 2 novel diffusion methods:
  - *Generalized Beltrami Flow*
  - *Tensor Total Variation*

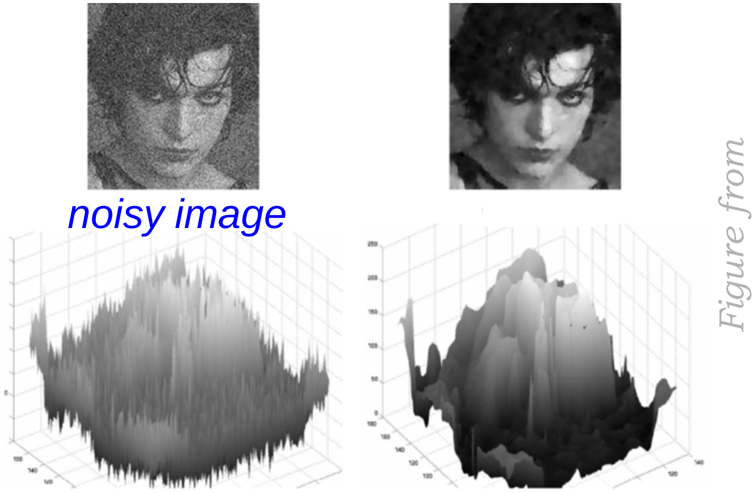
- These methods:
  - combine the advantages of variational and tensor-based diffusion approaches
  - yielded promising performance measures in denoising experiments

Generalization of the Beltrami Functional (1/2)

- Original Beltrami Flow
  [Sochen, Kimmel & Maladi, IEEE T-IP 98]
  - Interpretation of a vector-valued image *u* with *n* channels as a 2D surface embedded in $\mathbb{R}^{n+2}$:

$$(x,y) \longrightarrow (x\,,\,y\,,\,u_1(x,y)\,,\,u_2(x,y)\,,\,\dots\,u_n(x,y))$$

  - Flow towards the minimization of the surface area: tensor-based diffusion
  - It offers an elegant way to:
    - couple the image channels and
    - extend in the vector-valued case the properties of Total Variation
  - But, the diffusion tensor is not regularized (no neighborhood info)
    → limitations on the robustness to noise & edge enhancement



noisy image

embedded surface *instant from the flow*

example for the simplest case n=1

Figure from [Tschumperle, Thesis'02]

- To overcome these limitations, we generalize the Beltrami Functional …

Generalization of the Beltrami Functional (2/2)

- Proposed generalization of the Beltrami functional:
  - We use higher dimensional mappings of the form:

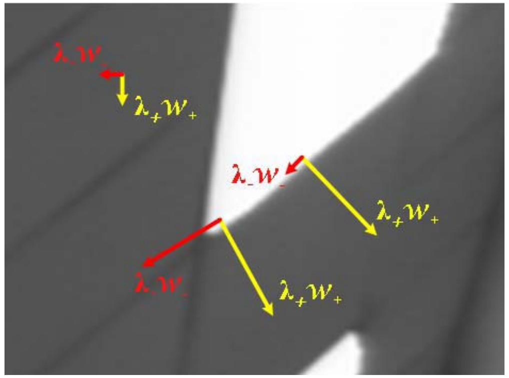
$$\boldsymbol{x} \longrightarrow (\boldsymbol{x}, \underbrace{\mathcal{P}^{\boldsymbol{u}}(\boldsymbol{x})})$$

image patch [Tschumperle & Brun, ICIP'09],
that contains weighted image values
*not only at point **x***
but also at points in a window around it

  - In this way, each ***x*** contributes to the area of the embedded surface by considering the image variation in its neighborhood
  - If the patch sampling step → 0, the area of the embedded surface tends to:

$$A[\boldsymbol{u}] = \int_{\Omega} \sqrt{(\alpha^2 + \lambda_1)(\alpha^2 + \lambda_2)}\mathrm{d}\boldsymbol{x}$$

    - $\lambda_i = \lambda_i(J_K(\nabla \boldsymbol{u}))$: eigenvalues of the structure tensor $J_K(\nabla \boldsymbol{u}) = K * \sum \nabla u_i \otimes \nabla u_i$



Generalized Functional based on the Structure Tensor

- $E[\boldsymbol{u}] = \int_\Omega \psi\,(\,\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\,)\,\mathrm{d}\boldsymbol{x}$
  - $\psi(\lambda_1, \lambda_2)$: cost function (increasing)
  - $J_K(\nabla \boldsymbol{u}) = K * \sum_{i=1}^{N} \nabla u_i \otimes \nabla u_i$: 2x2 structure tensor with:
    - eigenvalues λ_1, λ_2 , eigenvectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ (depend on *K*)
- Difficulty in the theoretical analysis:

  In contrast to most variational methods, Euler-Lagrange equations not applicable here
- Theorem: we have shown that the functional minimization leads to:

$$\begin{cases} \partial u_i / \partial t = \operatorname{div}\left(D_K \nabla u_i\right), \; i = 1, .., N\,, \\ D_K = K * \left(2\frac{\partial \psi}{\partial \lambda_1} \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_1 + 2\frac{\partial \psi}{\partial \lambda_2} \boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_2 \right) \end{cases}$$

novel general type of anisotropic diffusion

Tensor Total Variation

- 1^{st} special case of the novel generic functional:

$$E[\boldsymbol{u}] = \int_\Omega \psi\,(\,\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\,)\,\mathrm{d}\boldsymbol{x}$$

with $\psi(\lambda_1, \lambda_2) = \sqrt{\lambda_1} + \sqrt{\lambda_2}$

  - ❑ Steepest descent (applying the proved theorem):

$$\frac{\partial u_i}{\partial t} = \mathrm{div}\left(\left[K * \left(\frac{1}{\sqrt{\lambda_1}}\boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_1 + \frac{1}{\sqrt{\lambda_2}}\boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_2\right)\right]\nabla u_i\right)\;,\; i = 1, .., N$$

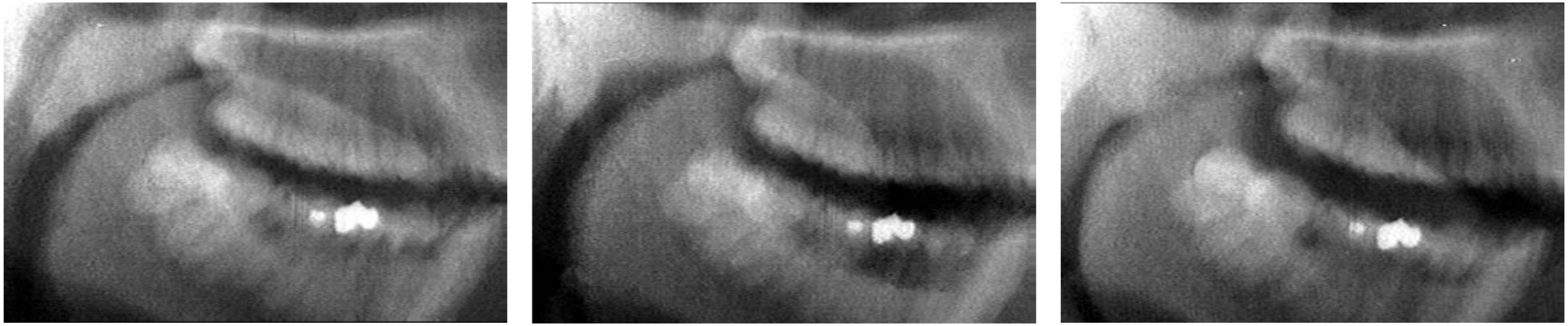
  - ❑ Classic TV: special sub-case with: N=1(graylevel images) and $K = \delta(\boldsymbol{x})$

(a) Noisy Input
(PSNR=20 dB)

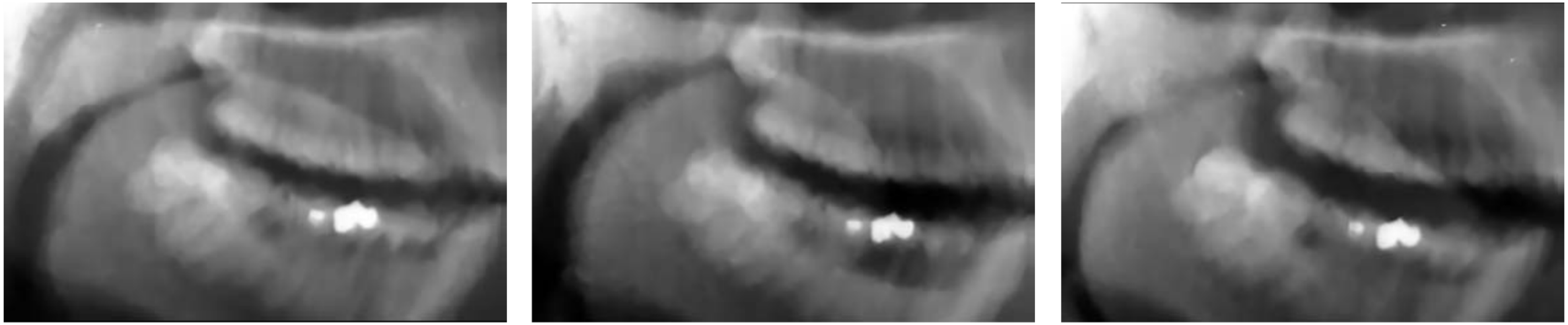
(b) TV PDE
(PSNR=26.5 dB, t=16.4)

(c) Tensor TV PDE
(PSNR=27.1 dB, t=9.6)

Tensor Total Variation: Example



Input sequence



Output sequence

Application of Tensor Total Variation in a sequence of X-ray images

Generalized Beltrami Flow

- 2nd special case of the novel generic functional:

$$E[\boldsymbol{u}] = \int_\Omega \psi\left(\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\right) \mathrm{d}\boldsymbol{x}$$

with $\psi(\lambda_1, \lambda_2) = \sqrt{(\alpha^2 + \lambda_1)(\alpha^2 + \lambda_2)}$

  - ❑ Steepest descent (applying the proved theorem):

$$\frac{\partial u_i}{\partial t} = \mathrm{div}\left(\left[K*\left(\sqrt{\frac{\alpha^2+\lambda_2}{\alpha^2+\lambda_1}}\boldsymbol{\theta}_1\otimes\boldsymbol{\theta}_1+\sqrt{\frac{\alpha^2+\lambda_1}{\alpha^2+\lambda_2}}\boldsymbol{\theta}_2\otimes\boldsymbol{\theta}_2\right)\right]\nabla u_i\right)$$

  - ❑ Classic Beltrami flow [Sochen et. al, IEEE T-IP 98]: special sub-case with $K = \delta(\boldsymbol{x})$ and minimization in the space of embeddings

(a) Noisy Input (PSNR=20 dB)

(b) Beltrami Flow (PSNR=23.4 dB)

(c) Gener. Beltrami Flow (PSNR=24.0 dB)

Other Interesting Special Cases

- Other special cases of the novel generic functional:

  $E[\boldsymbol{u}] = \int_\Omega \psi\left(\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\right) \mathrm{d}\boldsymbol{x}$ with:

  - $\psi(\lambda_1, \lambda_2) = \phi(\lambda_1 + \lambda_2)$: Steepest descent:

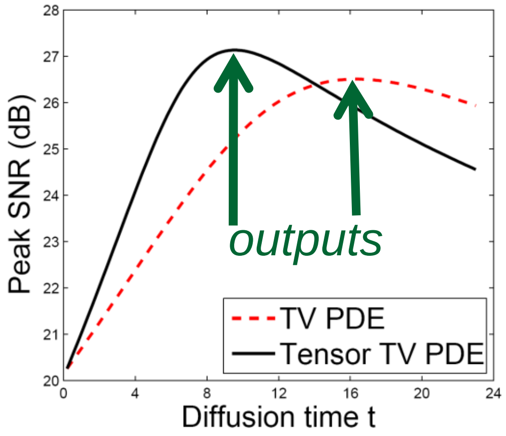
    $$\partial u_i / \partial t = \operatorname{div}\left(2\left[K * \varphi'(K * \|\nabla \boldsymbol{u}\|^2)\right] \nabla u_i\right)$$

    →novel regularization of the Perona-Malik model
    →regularization of Sapiro's Vectorial TV: $\psi = \sqrt{\lambda_1 + \lambda_2}$

  - $K = \delta(\boldsymbol{x})$ (no regularizing convolution):
    - Studied in [Blomgren & Chan T-IP'98, Tschumperlé & Deriche, T-PAMI'05]
    - The corresponding diffusion is anisotropic only if the image channels are $N \geq 2$
    - No incorporation of neighborhood info

Denoising Experiments: Framework

- Experimental Framework
  - take a noise-free reference image
  - add gaussian noise
  - input in the compared diffusion methods
  - compute PSNR during each PDE flow and output the image with the maximum PSNR



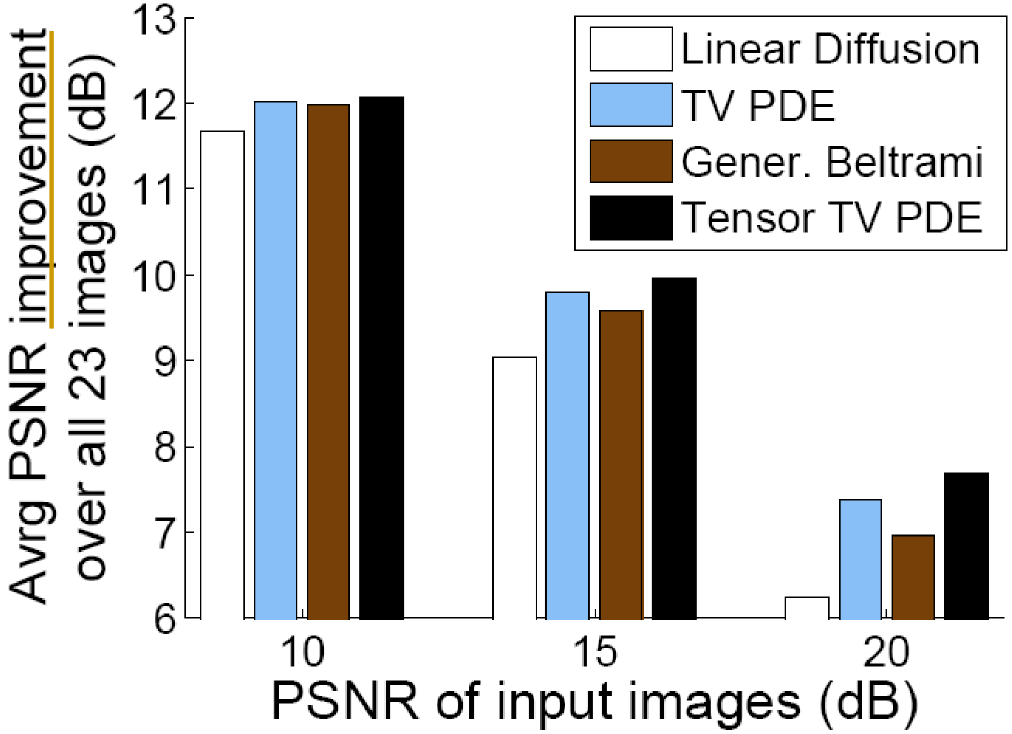
- This framework has been repeated for reference images from a dataset of *CIPR*: www.cipr.rpi.edu/resource/stills/kodak.html

23 natural images of size 768 x 512 pixels

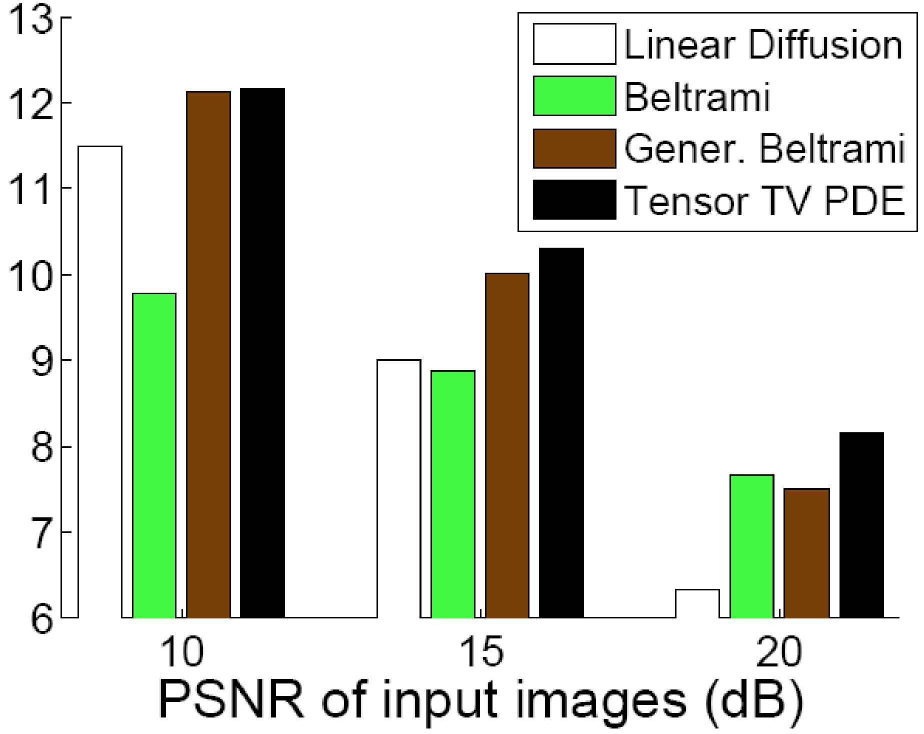
…

- Both graylevel & color versions of images have been used

Denoising Experiments: Performance Measures



(a) Graylevel images



(b) Color images

Summary & Conclusions

- We introduced a generic functional that
  - is based on the image structure tensor
  - generalizes Total Variation & Beltrami Functionals
- We proved that its minimization leads to a novel general type of anisotropic diffusion
- We proposed two novel anisotropic diffusion methods
- Several denoising experiments showed the potential of the novel approach

- The proposed framework opens various new directions for future research
  - Many other special cases of the generic functional might be promising
  - Thanks to the variational interpretation, such regularized tensor-based diffusions can be applied to other problems, e.g.:
    - image restoration, inpainting and interpolation

Thank you for your attention

For more information, demos, and current results:

http://cvsp.cs.ntua.gr