Sparse Approximate Solutions to Max-Plus Equations

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Overview

Motivation:

- Inverse problems: We observe a vector $\mathbf{b}$ as linear measurements of an unknown quantity $\mathbf{x}$ through a system $\mathbf{A}$. We want to recover the initial information. Problem: Possibly infinite candidates that explain the data.

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Core of the problem:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ . & . & . & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

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What is the solution with the least non zero elements? The most *sparse*?

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Max-plus Algebra

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$$b_1 = \max(a_{11} + x_1, a_{12} + x_2, \dots, a_{1n} + x_n),$$
$$b_2 = \max(a_{21} + x_1, a_{22} + x_2, \dots, a_{2n} + x_n)$$
$$\vdots$$

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What is the solution with the least number of non $-\infty$ elements? The most sparse solution?

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  - [Gaubert et. al. 2011] Pruning in optimal control - related to sparse ℓ_1 approximate solutions.
  - [Tsiamis & Maragos 2019] introduced the concept of sparsity in max-plus algebra.

- Values from $\mathbb{R} \cup \{-\infty\}$.
- Max-plus and min-plus products:

$$[\mathbf{A} \boxplus \mathbf{x}]_i \triangleq \bigvee_{k=1}^{n} a_{ik} + x_k, \; [\mathbf{A} \boxplus' \mathbf{x}]_i \triangleq \bigwedge_{k=1}^{n} a_{ik} + x_k$$

- A max-plus equation $\mathbf{A} \boxplus \mathbf{x} = \mathbf{b}$ has a solution iff $\hat{\mathbf{x}} = -\mathbf{A}^\intercal \boxplus' \mathbf{b}$ (*principal solution*) satisfies it.
- $\mathbf{A} \boxplus \hat{\mathbf{x}} \leq \mathbf{b}$.

Submodularity preliminaries (1/2)

Definition (Submodular)

A set function $f : 2^U \to \mathbb{R}$ is called *submodular* if $\forall A \subseteq B \subseteq U,\ k \notin B$ holds:

$$f(A \cup \{k\}) - f(A) \geq f(B \cup \{k\}) - f(B).$$

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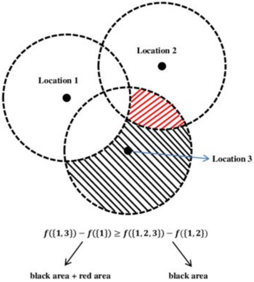

Figure: [Liu et al. 2019]

Generalization of submodularity

Definition (Submodularity ratio of an increasing, non-negative function [Das & Kempe 2018])

$$\gamma_{U,k}(f) \triangleq \min_{L \subseteq U, S: |S| \leq k, S \cap L = \emptyset} \frac{\sum_{x \in S} f(L \cup \{x\}) - f(L)}{f(L \cup S) - f(L)}$$

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Proposition

An increasing function $f : 2^U \to \mathbb{R}$ *is submodular if and only if* $\gamma_{U,k}(f) \geq 1,\ \forall\ U, k$.

Sparsity in Max-plus algebra

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The *support set* of a vector is the set of indices of its values that are not equal to $-\infty$, that is: $\mathrm{supp}(\mathbf{x}) = \{j \mid x_j \neq -\infty\}$.

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Theorem (Tsiamis & Maragos 2019)

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essentially: Minimum Set Cover.

Sparse Approximate Solutions formulation

Problem formulation

$$\arg\min_{x} |\text{supp}(\mathbf{x})|$$
$$\text{s.t. } \|\mathbf{b} - \mathbf{A} \boxplus \mathbf{x}\|_p^p \leq \epsilon,$$
$$\mathbf{A} \boxplus \mathbf{x} \leq \mathbf{b}, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{y} \in \mathbb{R}^m.$$

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Notes

- We restrict the ℓ_p, $p < \infty$, error to be small.
- We add an extra constraint $\mathbf{A} \boxplus \mathbf{x} \leq \mathbf{b}$.
- Observe that for $\epsilon = 0$ reduces to an $\mathcal{NP}$-complete problem.
- The case $p = 1$ was examined in [Tsiamis & Maragos 2019].

Sparsity as a discrete optimization problem

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New Problem formulation

$$\begin{aligned} \arg\min_{T} & \; |T| \\ \text{s.t. } & E_p(T) \leq \epsilon \end{aligned}$$

Theorem

Error function E_p is decreasing and supermodular.

Sparsity as a tractable discrete optimization problem

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Notes

- Proof leverages the submodularity ratio which clarifies the analysis.
- Problem becomes: Cardinality minimization problem subject to a supermodular equality constraint $\Rightarrow$ Fast greedy algorithm works!

Algorithm 1: Approximate solution of problem (1)

Input: $\mathbf{A}, \mathbf{b}$

Compute $\hat{\mathbf{x}} = (-\mathbf{A})^\intercal \boxplus' \mathbf{b}$

if $E_p(J) > \epsilon$ **then**

  return Infeasible

Set $T_0 = \emptyset$, $k = 0$

while $E_p(T_k) > \epsilon$ **do**

  $j = \arg\min_{s \in J \setminus T_k} E_p(T_k \cup \{s\})$

  $T_{k+1} = T_k \cup \{j\}$

  $k = k + 1$

end

$x_j = \hat{x}_j, j \in T_k$ and $x_j = -\infty$, otherwise

return x, T_k

Time complexity: $\mathcal{O}(nm + n^2)$

Approximation ratio: $\mathcal{O}(\log(m\Delta^p))$, $\Delta = \bigvee_{i,j}(b_i - A_{ij} - \hat{x}_j)$.

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Searching only for approximate sub-solutions is restrictive. Can we overcome this?

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Proposition

If $\mathbf{x}_{p,\epsilon}$ is a solution of ℓ_p Problem (1), then $\mathbf{x}^* = \mathbf{x}_{p,\epsilon} + \frac{\|b - A\boxplus x_{p,\epsilon}\|_\infty}{2}$ has the *smallest* ℓ_∞ error over **all** sparse vectors with the same support set T.

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Computational overhead: $\mathcal{O}(m|T|)$

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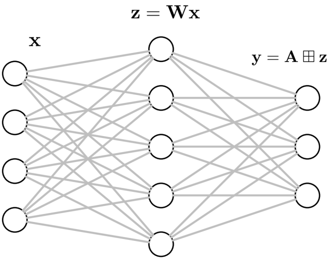

Neuron pruning as a sparsity problem

Consider a simple two layer network that performs a linear transformation followed by a dilation (*max-plus block* [Zhang et al. 2019]):

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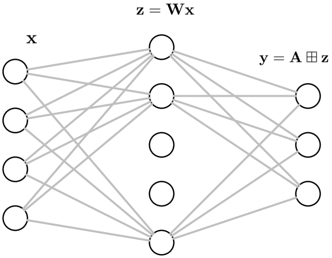

- 2 networks of 64 and 128 neurons, trained for 20 epochs, with SGD.
- We are able to find the 10 most important neurons automatically and prune the rest of them (recording same accuracy).

	MNIST		FashionMNIST	
	64	128	64	128
Full model	92.21	92.17	79.27	83.37
Pruned ($n = 10$)	92.21	92.17	79.27	83.37

Table: Test set accuracy before and after pruning.

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Thank you for your attention!