# A PDE APPROACH TO NONLINEAR IMAGE SIMPLIFICATION VIA LEVELINGS AND RECONSTRUCTION FILTERS

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## ABSTRACT

We present a nonlinear partial differential equation (PDE) that models the generation of a large class of advanced morphological filters, the levelings and the openings/closings by reconstruction. These types of filters are very useful in numerous image analysis and vision tasks ranging from enhancement, feature detection, image simplification, to segmentation. The developed PDE models these nonlinear filters as the limit of a controlled growth starting from an initial seed signal. This growth is of the multiscale dilation or erosion type and the controlling mechanism is a switch that reverses the growth when the difference between the current evolution and a reference signal switches signs. We discuss theoretical aspects of this PDE, propose a discrete algorithm for its numerical solution and corresponding filter implementation, and provide insights via several experiments. Finally, we outline its use for improving the Gaussian scale-space by using the latter as initial seed to generate multiscale levelings that have a superior preservation of image edges and boundaries.

## 1. INTRODUCTION

In computer vision there have been proposed continuous models for scale-space image analysis based on partial differential equations (PDEs). Motivations for using PDEs include better and more intuitive mathematical modeling, connections with physics, better isotropy, better approximation to the Euclidean geometry of the problem, and subpixel accuracy. While many such continuous approaches have been linear (the most notable example being the isotropic heat diffusion PDE for modeling the Gaussian scale-space), many among the most useful ones are nonlinear. Areas where there is a need to develop nonlinear approaches include the class of problems related to scale-space analysis and multiscale image smoothing. In contrast to the shifting and blurring of image edges caused by linear smoothers, there is a large variety of nonlinear smoothers that either suffer less from or completely avoid these shortcomings. Simple examples are the classic morphological openings and closings (cascades of erosions and dilations) as well as the median filters. The openings suppress signals peaks, the closings eliminate valleys, whereas the medians have a more symmetric behavior. All three filter types preserve well vertical image edges but may shift and blur horizontal edges. A much more powerful class of filters are the *reconstruction* openings and closings which, starting from a *reference* signal $f$ consisting of several parts and a *marker* (initial seed) $g$ inside some of these parts, can reconstruct whole objects with exact preservation of their boundaries and edges. In this reconstruction process they simplify the original image by completely eliminating smaller objects inside which the marker cannot fit. The reconstruction filters enlarge the flat zones of the image [6]. One of their disadvantages is that they treat asymmetrically the image foreground and background. A recent solution to this asymmetry problem came from the development of a more general powerful class of morphological filters, the levelings [3], which include as special cases the reconstruction openings and closings. They are transformations $\Lambda(f, g)$ that depend on two signals, the reference $f$ and the marker $g$. Reconstruction filters and levelings have found numerous applications in a large variety of problems involving image enhancement and simplification, geometric feature detection, and segmentation. They also possess many useful algebraic and scale-space properties, as discussed in [4].

In this paper we develop a PDE that can model and generate levelings. This PDE works by growing a marker signal $g$ in a way that the growth extent is controlled by a reference signal $f$ and its type (expansion or shrinking growth) is switched by the sign of the difference between $f$ and the current evolution. This growth is modeled by PDEs that can generate multiscale dilations or erosions. Therefore, we start first with a brief background section on dilation PDEs. After-

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wards, we introduce a PDE for levelings of 2D images, propose a discrete numerical algorithm for its implementation, and provide insights via experiments. Finally, we outline the use of these PDEs for improving the Gaussian scale-space by using the latter as initial seed to generate multiscale levelings that have a superior preservation of image edges and boundaries.

## 2. DILATION AND EROSION PDES

Inspired by the use in the computer vision community of the classic heat PDE to model the linear Gaussian scale-space, in 1992 three teams of researchers [1],[2],[7]) independently published nonlinear PDEs that model the nonlinear scale-space of elementary morphological operators; each team focused on different aspects of the problem.

Multiscale flat dilations and erosions of a 2D signal $f(x, y)$ by scaled versions of a flat structuring element, which is a unit-scale compact convex and symmetric planar set $B$, are represented by the space-scale functions $\delta(x, y, t)$ and $\varepsilon(x, y, t), t > 0$:

$$\begin{aligned} \delta(x,y,t) &\equiv (f \oplus tB)(x,y) = \sup_{(a,b)\in tB} f(x-a, y-b) \\ \varepsilon(x,y,t) &\equiv (f \ominus tB)(x,y) = \inf_{(a,b)\in tB} f(x+a, y+b) \end{aligned}$$

where $tB = \{(ta, tb) : (a, b) \in B\}$. If $B$ is the unit disk, then the PDEs generating the corresponding multiscale circular dilation and erosion of $f$ are:

$$\delta_t = ||\nabla \delta|| = \sqrt{(\delta_x)^2 + (\delta_y)^2}; \quad , \quad \varepsilon_t = -||\nabla \varepsilon||$$

with initial values $\delta(x, y, 0) = \varepsilon(x, y, 0) = f(x, y)$.

## 3. PDE FOR LEVELINGS

Consider a 2D signal $f(x, y)$ and a marker signal $g(x, y)$ from which a leveling $\Lambda(f, g)$ will be produced.

If $g \le f$ everywhere and we start iteratively growing $g$ via incremental flat dilations with a disk of an infinitesimally small radius $\Delta t$ but without ever growing the result above the graph of $f$, then in the limit we shall have produced the *opening by reconstruction* of $f$ (with respect to the marker $g$), which is a special leveling. The infinitesimal generator of this signal evolution can be modeled via a dilation PDE that has a mechanism to stop the growth whenever the intermediate result attempts to create a function larger than $f$. Specifically, let $u(x, y, t)$ represent the evolutions of $f$ with initial value $u_0(x, y) = u(x, y, 0) = g(x, y)$. Then, $u$ is a weak solution of the following initial-value PDE system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \text{sign}(f-u)\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} \quad (1) \\ u(x,y,0) &= g(x,y) \end{aligned}$$

where $\text{sign}(r)$ is equal to $+1$ if $r > 0$, $-1$ if $r < 0$ and 0 if $r = 0$. Since $g \le f$, the above initial-value PDE system models a *conditional dilation* that grows the intermediate result as long as it does not exceed $f$. In the limit we obtain the final result $u_\infty(x, y) = \lim_{t\to\infty} u(x, y, t)$. The mapping $u_0 \mapsto u_\infty$ is the opening by reconstruction filter.

If in the above paradigm we reverse the order between $f$ and $g$, i.e., assume that $g \ge f$, and replace the positive growth (dilation) of $g$ with negative growth via erosion that stops when the intermediate result attempts to become smaller than $f$, then we obtain the *closing by reconstruction* of $f$ with respect to the marker $g$. This is another special case of a leveling, whose generation can also be modeled by the same PDE (1) but with a marker that exceeds $f$. This dynamical system models a *conditional erosion* that keeps reducing the intermediate result as long as it does not decrease below $f$.

What happens if we use any of the above PDE when there is no specific order between $f$ and $g$? In such a case the PDE (1) has a varying coefficient $\text{sign}(f-u)$ with spatio-temporal dependence which controls the instantaneous growth and stops it whenever $f = u$. (Of course, there is no growth also at stationary points where $\nabla u = 0$.) The control mechanism is of a switching type: For each $t$, at pixels $(x, y)$ where $u(x, y, t) < f(x, y)$ it acts as a dilation PDE and hence shifts outwards the surface of $u(x, y, t)$ but does not move the extrema points. Wherever $u(x, y, t) > f(x, y)$ the PDE acts as an erosion PDE and reverses the direction of propagation. The final result $u_\infty(x, y) = \lim_{t\to\infty} u(x, y, t)$ is a general *leveling* of $f$ with respect to $g$. We call (1) a *switched dilation* PDE. The switching action of this PDE model occurs at zero crossings of $f - u$ where shocks are developed. Obviously, the PDEs generating the opening and closing by reconstruction are special cases where $g \le f$ and $g \ge f$, respectively. However, the PDEs generating the reconstruction filters do not involve switching of growth.

The switching between a dilation- or erosion-type PDE also occurs in a class of nonlinear time-dependent PDEs which was proposed in [5] to deblur images and/or enhance their contrast by generating shocks and hence sharpening edges. For 2D images a special case of such a PDE is

$$u_t = -||\nabla u||\text{sign}(\nabla^2 u) \quad (2)$$

A major conceptual difference between the above edge sharpening PDE and our PDE generating levelings is that in the former the switching is determined by the edges, i.e., the inflection points of $u$ itself whereas in the latter the switching is controlled by comparing $u$ against the external reference signal $f$.

To produce a shock-capturing and entropy-satisfying numerical method for solving the general leveling PDE (1), we use ideas from the technology of solving PDEs corresponding to hyperbolic conservation laws and Hamilton-Jacobi

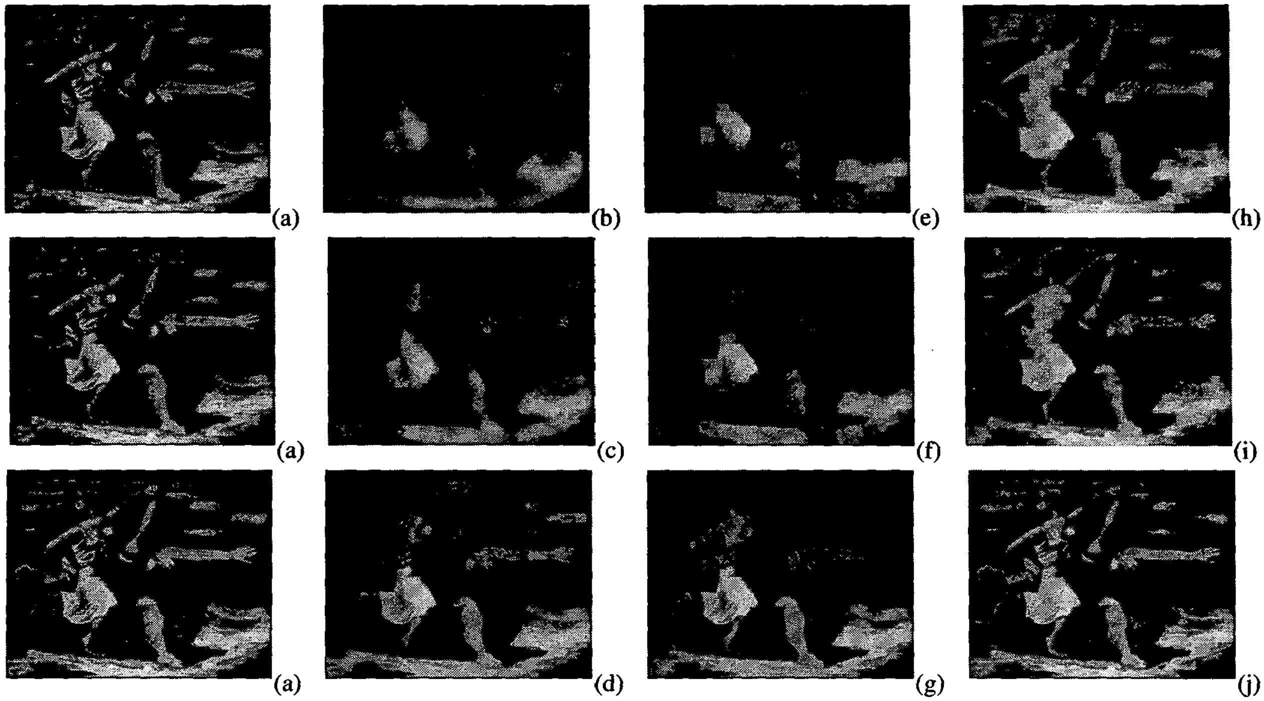


**Fig. 1**. Evolutions of the 2D leveling PDE on the reference image (a) using 3 markers. Columns show evolutions from the same marker. First row shows markers ($t = 0$), second row shows evolutions at $t = 10\Delta t$, and on third row the final levelings (after convergence). In column (b,c,d), the marker (b) was obtained from a 2D convolution of $f$ with a Gaussian of $\sigma = 4$. In column (e,f,g), the marker (e) was an opening by a square of $9 \times 9$ pixels and hence the corresponding leveling (g) is a reconstruction opening. In column (h,i,j), the marker (h) was a closing by a square of $9 \times 9$ pixels and hence the corresponding leveling (j) is a reconstruction closing. ($\Delta x = \Delta y = 1$, $\Delta t = 0.25$.)

formulations. Thus, we propose the following discretization sheme, which is an adaptation of a scheme proposed in [5] for solving (2).

Let $U^n_{i,j}$ be the approximation of $u(x, y, t)$ on a grid $(i\Delta x, j\Delta y, n\Delta t)$. Consider the spatial forward and backward differences (as functions of $U, i, j, n$):

$$D_{\pm x} \equiv \pm \frac{U^n_{i\pm1,j} - U^n_{i,j}}{\Delta x}, \quad D_{\pm y} \equiv \pm \frac{U^n_{i,j\pm1} - U^n_{i,j}}{\Delta y}$$

Then we approximate the leveling PDE (1) by the following nonlinear difference equation:

$$\begin{aligned} U^{n+1}_{i,j} &= U^n_{i,j} - \Delta t[\cdots \\ &(S^n_{i,j})^+ \sqrt{(D_{-x}{}^+)^2 + (D_{+x}{}^-)^2 + (D_{-y}{}^+)^2 + (D_{+y}{}^-)^2} \\ &+ (S^n_{i,j})^- \sqrt{(D_{+x}{}^+)^2 + (D_{-x}{}^-)^2 + (D_{+y}{}^+)^2 + (D_{-y}{}^-)^2}\ ] \end{aligned} \quad (3)$$

where $S^n_{i,j} = \text{sign}(f(i\Delta x, j\Delta y) - U^n_{i,j})$, and we denote $(r)^+ = \max(r, 0)$, $(r)^- = \min(r, 0)$ for any real $r$. For stability, $(\Delta t/\Delta x + \Delta t/\Delta y) \leq 0.5$ is required. Further, at each iteration we enforce the sign consistency

$$\text{sign}(U^n - f) = \text{sign}(g - f) \quad (4)$$

We have not proved theoretically that the above iterated scheme converges when $n \to \infty$, but through many experiments we have observed that it converges in a finite number of steps. Three examples of the action of the above 2D algorithm are shown in Fig. 1.

## 4. FROM GAUSSIAN SCALE-SPACE TO MULTISCALE LEVELINGS

Consider a reference signal $f$ and a leveling $\Lambda$. If we can produce various markers $g_i$, $i = 1, 2, 3, \ldots$, that are related to some increasing scale parameter $i$ and produce the levelings of $f$ with respect to these markers, then we can generate *multiscale* levelings in some approximate sense. This scenario will be endowed with an important property if we slightly change it to the following hierarchy:

$$\ell_1 = \Lambda(f, g_1),\ \ell_2 = \Lambda(\ell_1, g_2),\ \ell_3 = \Lambda(\ell_2, g_3), \ldots \quad (5)$$

The above sequence of steps insures that $\ell_j$ is a leveling of $\ell_i$ for $j > i$.

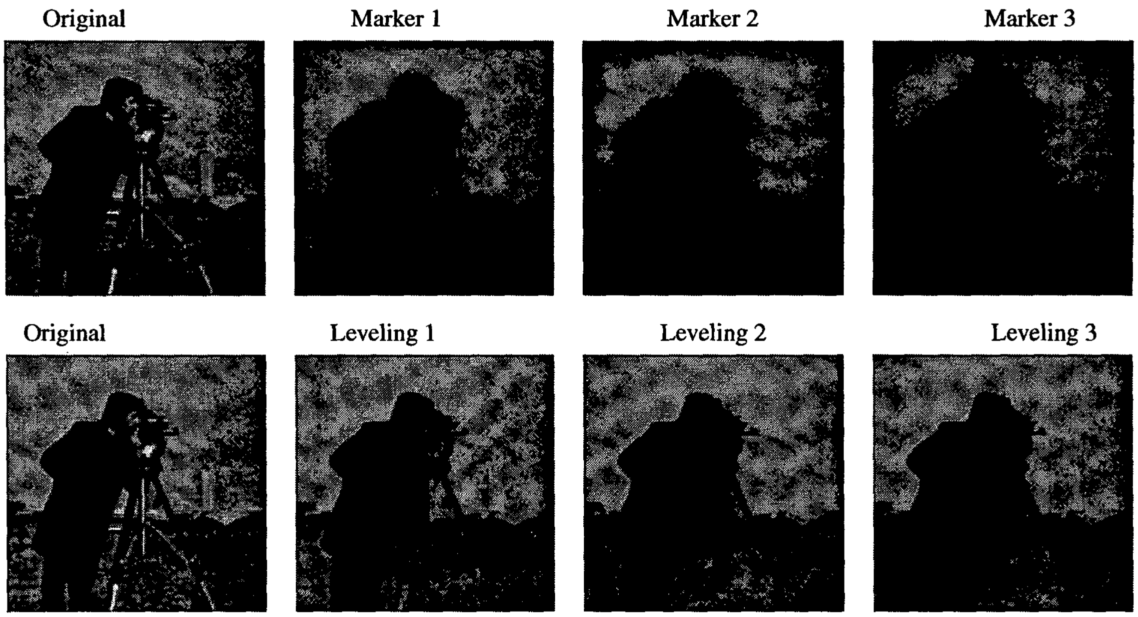


**Fig. 2.** Multiscale image levelings. The markers were obtained by convolving reference image with 2D Gaussians of standard deviations $\sigma = 3, 5, 7$. (The levelings were produced by running the leveling PDE with $\Delta x = \Delta y = 1$, $\Delta t = 0.25$.)

The sequence of markers $g_i$ may be obtained from $f$ in any meaningful way. A particularly interesting choice we have considered is the case where the $g_i$ are multiscale convolutions of $f$ with Gaussians of increasing standard deviations $\sigma_i$. Examples of constructing multiscale levelings from Gaussian convolution markers according to (5) are shown in Fig. 2 for an image $f$. The sequence of the multiscale markers can be viewed as a scale-sampled Gaussian scale-space. As shown in the experiments, the image edges and boundaries which have been blurred and shifted by the Gaussian scale-space are better preserved across scales by the multiscale levelings that use the Gaussian convolutions as markers. Thus, several computer vision applications that employ the Gaussian scale-space may benefit by using the Gaussian scale-space as a first phase and the above multiscale leveling scheme as a second phase that sharpens the Gaussian convolutions towards the original image.

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