A VARIATIONAL FORMULATION OF PDE'S FOR DILATIONS AND LEVELINGS

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Abstract Partial differential equations (PDEs) have become very useful modeling and computational tools for many problems in image processing and computer vision related to multiscale analysis and optimization using variational calculus. In previous works, the basic continuous-scale morphological operators have been modeled by nonlinear geometric evolution PDEs. However, these lacked a variational interpretation. In this paper we contribute such a variational formulation and show that the PDEs generating multiscale dilations and erosions can be derived as gradient flows of variational problems with nonlinear constraints. We also extend the variational approach to more advanced object-oriented morphological filters by showing that levelings and the PDE that generates them result from minimizing a mean absolute error functional with local sup-inf constraints.

Keywords: scale-spaces, PDEs, variational methods, morphology.

1. Introduction

Partial differential equations have a become a powerful set of tools in image processing and computer vision for modeling numerous problems that are related to multiscale analysis. They need continuous mahematics such as differential geometry and variational calculus and can benefit from concepts inspired by mathematical physics. The most investigated partial differential equation (PDE) in imaging and vision is the linear isotropic heat diffusion PDE because it can model the Gaussian scale-space, i.e. its solution holds all multiscale linear convolutions of an initial image with Gaussians whose scale parameter is proportional to their variance. In addition, to its scale-space interpretation, the linear heat PDE can also be derived from a variational problem. Specifically, if we attempt to evolve an initial image into a smoother version by minimizing the L_2 norm of the gradient magnitude, then the PDE that results as the gradient descent flow to reach the minimizer is identical to the linear heat PDE. All

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the above ideas are well-known and can be found in numerous books dealing with classic aspects of PDEs and variational calculus both from the viewpoint of mathematical physics, e.g. [5], as well as from the viewpoint of image analysis, e.g. [12, 6, 14].

In the early 1990s, inspired by the modeling of the Gaussian scale-space via the linear heat diffusion PDE, three teams of researchers (Alvarez, Guichard, Lions & Morel [1], Brockett & Maragos [3, 4], and Boomgaard & Smeulders [19]) independently published nonlinear PDEs that model various morphological scale-spaces. Refinements of the above works for PDEs modeling multiscale morphology followed in [8, 7, 6]. However, in none of the previous works the PDEs modeling morphological scale-spaces were also given a direct variational interpretation. There have been only two indirect exceptions: i) Heijmans & Maragos [7] unified the morphological PDEs using Legendre-Fenchel 'slope' transforms, which are related to Hamilton-Jacobi theory and this in turn is related to variational calculus. ii) Inspired by the level sets methodology [13], it has been shown in [2, 15] that binary image dilations or erosions can be modeled as curve evolution with constant (± 1) normal speed. The PDE of this curve evolution results as the gradient flow for evolving the curve by maximizing or minimizing the rate of change of the enclosed area; e.g. see [17] where volumetric extensions of this idea are also derived. Our work herein is closer to [17].

In this paper we contribute a new formulation and interpretation of the PDEs modeling multiscale dilations and erosions by showing that they result as gradient flows of optimization problems where the volume under the graph of the image is maximized or minimized subject to some nonlinear constraints. Further, we extend this new variational interpretation to more complex morphological filters that are based on global constraints, such as the levelings [10, 11, 9].

2. Background

Variational Calculus and Scale-Spaces

A standard variational problem is to find a function $u = u(x, y)$ that minimizes the 'energy' functional

$$J[u] = \int\int F(x, y, u, u_x, u_y) dxdy \tag{1}$$

usually subject to natural boundary conditions, where F is a second-order continuously differentiable function. A necessary condition satisfied by an extremal function u is the Euler-Langange PDE $[F]_u = 0$, where $[F]_u$ is the Euler (variational) derivative of F w.r.t. u. In general, to reach the extremal function that minimizes J, we can set up a gradient steepest descent procedure starting from an initial function $u_0(x,y)$ and evolving it into a function $u(x,y,t)$, where t is an artificial marching parameter, that satisfies the evolution PDE

$$\frac{\partial u}{\partial t} = -[F]_u, \quad [F]_u = F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} \tag{2}$$

This PDE is called the *gradient flow* corresponding to the original variational problem. In some cases, as $t \to \infty$ the gradient flow will reach the minimizer of J. If we wish to *maximize* J, the corresponding gradient flow is $u_t = [F]_u$. (Short notation for PDEs: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $\nabla u = (u_x, u_y)$, $\nabla^2 u = u_{xx} + u_{yy}$.)

In the gradient flow formulation the evolving function $u = u(x,y,t)$ is a family of functions depending on the time parameter t and hence $J[u] = J(t)$. Then [5]

$$\frac{d}{dt} J[u] = \int\int u_t [F]_u dxdy \tag{3}$$

Thus, we can also view the Euler derivative $[F]_u$ as the *gradient of the functional* $J[u]$ *in function space*. This implies that, in getting from an arbitrary u_0 to the extremal, the PDE (2) of the gradient flow provides us with the fastest possible rate of decreasing J.

In scale-space analysis, we also start from an initial image $u_o(x,y)$ and evolve it into a function $u(x,y,t)$ with $u(x,y,0) = u_0(x,y)$. The mapping $u_0 \mapsto u$ is generated by some multiscale filtering at scale $t \geq 0$ or by some PDE. The PDEs of several known scale-spaces (e.g. the Gaussian) have a variational interpretation since they can be derived as gradient flows of functional minimization problems where the marching time t coincides with the scale parameter. For example, if $F = (1/2)||\nabla u||^2$, the gradient flow corresponding to minimizing $J = \int\int F$ is the isotropic heat diffusion PDE $u_t = \nabla^2 u$.

PDEs for Dilation/Erosion Scale-Spaces

Let $k : \mathbb{R}^m \to \overline{\mathbb{R}}$, $m = 1, 2, ...$, be a unit-scale upper-semicontinuous concave structuring function. Let $k_t(x) = tk(x/t)$ be its multiscale version, where both its values and its support have been scaled by a parameter $t \geq 0$. The *multiscale* Minkowski dilation $\oplus$ and erosion $\ominus$ of $f : \mathbb{R}^m \to \mathbb{R}$ by k_t are defined as the scale-space functions $\delta(x,t) = (f \oplus k_t)(x)$ and $\varepsilon(x,t) = (f \ominus k_t)(x)$:

$$\delta(x,t) = \bigvee_{y \in \mathbb{R}^m} f(y) + k_t(x-y), \quad \varepsilon(x,t) = \bigwedge_{y \in \mathbb{R}^m} f(y) - k_t(y-x),$$

where $\bigvee$ and $\bigwedge$ denote supremum and infimum, $\delta(x,0) = \varepsilon(x,0) = f(x)$. If $k(x,y)$ is *flat*, i.e. equal to 0 at points $(x,y) \in B$ and $-\infty$ else, where B is a unit disk, the PDEs generating the multiscale *flat* dilations and erosions of 2D images $f(x,y)$ by a disk B are [1, 4, 19]

$$\delta_t = ||\nabla\delta|| = \sqrt{(\delta_x)^2 + (\delta_y)^2}, \quad \varepsilon_t = -||\nabla\varepsilon|| \tag{4}$$

For 1D signals $f(x)$, B becomes the interval $[-1, 1]$ and the above PDEs become [4]

$$\delta_t = |\delta_x|, \quad \varepsilon_t = -|\varepsilon_x| \tag{5}$$

If k is the compact-support spherical function, i.e. $k(x,y) = (1 - x^2 - y^2)^{1/2}$ for $x^2 + y^2 \leq 1$ and $-\infty$ else, the PDE generating these spherical dilations is [4]

$$\delta_t = \sqrt{1 + (\delta_x)^2 + (\delta_y)^2}. \tag{6}$$

3. Variational Approach for Dilation PDEs

Let $u_0(x,y)$ be some smooth initial image over a rectangular image domain R with zero values outside R. Without loss of generality, we can assume that $u_0(x,y) \geq 0$ over R; otherwise, we consider as initial image the function $u_0 - \bigwedge u_0$. Let $u(x,y,t)$ be some scale-space analysis with $u(x,y,0) = u_0(x,y)$ that results from growing u_0 via dilation of the hypograph (umbra) of u_0 by some 3D structuring element $tB = \{tb : b \in B\}$ of radius $t \geq 0$, where $B \subseteq \mathbb{R}^3$ is a unit-radius compact symmetric convex set. From mathematical morphology we know that this 3D propagation of the graph of u_0 corresponds to a function dilation,

$$u(x,y,t) = u_0(x,y) \oplus k_t(x,y), \quad k_t(x,y) = \sup\{v : (x,y,v) \in tB\}, \tag{7}$$

of u_0 by a structuring function k_t that is the upper envelope of tB. We shall study three special cases of B: 1) a vertical line segment B_v, 2) a horizontal disk B_h, and 3) a sphere B_n. From (7), the three corresponding dilation scale-spaces are:

$$\begin{array}{ll} B = \text{v.line:} & u(x,y,t) = u_0(x,y) + t \\ B = \text{disk:} & u(x,y,t) = \bigvee_{||(a,b)|| \leq t} u_0(x-a, y-b) \\ B = \text{sphere:} & u(x,y,t) = \bigvee_{||(a,b)|| \leq t} u_0(x-a, y-b) + t\sqrt{1 - (\frac{a}{t})^2 - (\frac{b}{t})^2} \end{array} \tag{8}$$

While we know the gererating PDEs for the above scale-spaces (see [4]), in this paper our goal is to provide a variational interpretation for these PDEs and their solutions in (7). Define the multiscale *volume* functional

$$V(t) = \int\int u(x,y,t)dxdy = \int\int_{R(t)} u(x,y,t)dxdy \tag{9}$$

where $R(t)$ is the Minkowski dilation of the initial rectangular domain R with the projection of tB onto the plane. We wish to find the PDE generating u by creating a gradient flow that maximizes the rate of growth of $V(t)$. The classic approach [5] is to consider the time derivative $\dot{V}(t) = dV/dt$ as in (3). However, this is valid only when u is allowed to vary by remaining a function, e.g. $u \to u + \Delta t g$ where g is a perturbation function. Thus, u is allowed to vary in function space along a ray in the 'direction' of g. However, in our problem we have such a case only when $B = B_v$. In the other two cases u evolves as a graph by dilating its surface with a 3D set $\Delta t B$. To proceed, we convert the problem to a more usual variational formulation (i) by modeling the propagation of the graph of u as the evolution of a multiscale parameterized closed surface $\vec{S}(q_1, q_2, t)$, and (ii) by expressing the volume V as a surface integral around this closed surface. A similar approach as in step (i) has also been used in [18] for geometric flows of images embedded as surfaces in higher-dimensional spaces.

We start our discussion from a simpler (but conceptually the same as above) case where $u_0 = u_0(x)$ is a *1D nonnegative image* with nonzero values over an interval R. Let $u(x,t) = u_0(x) \oplus k_t(x)$ be the multiscale dilation of u_0 by a structuring function k_t that is the upper envelope of a 2D set tB where B is a 2D version of the previous 3D unit-radius symmetric convex set; i.e., B is either 1) a vertical line segment B_v, or 2) a horizontal line segment B_h, or 3) a disk B_n. First, we model the propagation of the graph of u as the evolution of a multiscale parameterized curve $\vec{C}(q,t) = (x(q,t), y(q,t))$, whose top part is the graph of u traced when $q \in R(t)^s$ and whose bottom part is the interval $R(t)$ traced when $q \in R(t)$ [where $R^s = \{-q : q \in R\}$]. This implies

$$\begin{array}{llll} y(q,t) = u(x,t), & x_q = -1,\ y_q = -u_x, & q \in R(t)^s \\ y(q,t) = 0, & x_q = 1, & q \in R(t) \end{array} \tag{10}$$

where subscripts denote partial derivatives. Then, we consider the *area* $A(t)$ under u and express it (using Green's theorem) as a line integral around this closed curve:

$$A(t) = \int u(x,t)dx = \frac{1}{2}\int_{C(t)} (xy_q - yx_q)dq = \frac{1}{2}\int_0^{L_c(t)} < \vec{C}, \vec{N} > ds \tag{11}$$

where s is arclength, $< \cdot >$ denotes inner product, $\vec{N}$ is the outward unit normal vector of the curve, $L_c(t) = L(t) + \mathrm{Len}(R(t))$ is the length of the closed curve $C(t)$, and $L(t) = \int_{R(t)} \sqrt{1+u_x^2}dx$ is the length of the graph of u. Next follows our first main result.

Theorem 1 *Maximization of the area functional $A(t)$ when the graph of $u(x,t)$ is dilated by tB with unit curve speed, where B is any of the following unit-radius 2D summetric convex sets, has a gradient flow governed by the* *following corresponding PDEs:*

$$B = \text{vert.line} \quad \Longrightarrow \quad u_t = 1 \tag{12}$$

$$B = \text{horiz.line} \quad \Longrightarrow \quad u_t = |u_x| \tag{13}$$

$$B = \text{disk} \quad \Longrightarrow \quad u_t = \sqrt{1 + |u_x|^2} \tag{14}$$

with $u(x, 0) = u_0(x)$.

Proof: Since we evolve u toward increasing $A(t)$, the graph curve speed $\vec{C}_t(q, t)$ must point outward for all $q \in R(t)^s$, i.e. $< \vec{C}_t, \vec{N} > \geq 0$. By (10), we can write the area functional as

$$A(t) = \int_{R(t)} u dx = \frac{1}{2} \int_0^{L(t)} < \vec{C}, \vec{N} > ds + \text{Len}[R(t)] \tag{15}$$

Differentiating (15) w.r.t. t yields

$$\frac{d}{dt} A(t) = \int_0^{L(t)} < \vec{C}_t, \vec{N} > ds + \text{const} \tag{16}$$

where $\text{const} = d\text{Len}[R(t)]/dt$. When B is the disk, the velocity $\vec{C}_t$ is allowed any direction and hence selecting $\vec{C}_t = \vec{N}$ guarantees that $\dot{A}(t)$ assumes a maximum value (i.e. the flow has a direction in function space in which $A(t)$ is increasing most rapidly). When B is the vertical line, $\vec{C}_t$ must have only a constant vertical component. When B is the horizontal line, $\vec{C}_t$ must have only a horizontal component with value ± 1 according to the sign of u_x. Thus, the three choices for structuring element B induce the following curve velocities:

$$\begin{array}{lll} B = \text{vert.line} & \Longrightarrow & \vec{C}_t = (x_t, y_t) = (0, 1) \\ B = \text{horiz.line} & \Longrightarrow & \vec{C}_t = (x_t, y_t) = (\text{sgn}(-u_x), 0) \\ B = \text{disk} & \Longrightarrow & \vec{C}_t = (x_t, y_t) = \vec{N} = \frac{(-u_x, 1)}{\sqrt{1+u_x^2}} \end{array} \tag{17}$$

In all three cases we shall use the relation

$$u_t = y_t - u_x x_t \tag{18}$$

which follows from $y(q, t) = u(x, t)$. When B is the vertical line, we have $x_t = 0$ and $y_t = 1$. Hence, $u_t = 1$ which proves (12). When B is the horizontal line, $y_t = 0$ and $x_t = \text{sgn}(-u_x)$ which yields $u_t = |u_x|$ and proves (13). When B is the disk, we have $x_t = -u_x/v$ and $y_t = 1/v$ where $v = \sqrt{1 + u_x^2}$. This and (18) yield $u_t = 1/v + u_x^2/v = v$, which proves (14). □

The volumetric extension of the above ideas to the case of a 2D nonnegative image $u_0(x, y)$ whose graph surface is dilated by 3D sets tB to give the graph of a scale-space function $u(x, y, t)$ is conceptually straightforward. First, we model the boundary of the *ordinate set* of u (i.e. the part of the umbra of u lying above the planar domain R) as a multiscale parameterized closed surface $\vec{S}(q_1, q_2, t)$, where (q_1, q_2) parameterize the surface as the two local coordinates and are related to (x, y). The top part of this closed surface is the graph of u and the bottom part is the planar domain $R(t)$ of u. Second, we express the volume $V(t)$ and its derivative as a surface integral around this closed parameterized surface:

$$V(t) = \frac{1}{3}\int < \vec{S}, \vec{N} > d\vec{S}, \quad \frac{d}{dt}V(t) = \int < \vec{S}_t, \vec{N} > d\vec{S} \tag{19}$$

For arbitrary 3D shapes enclosed by a surface, the above formulas were used in [17] to derive volume minimizing flows for shape segmentation.

Theorem 2 *Maximization of the volume functional $V(t)$ when the graph surface of $u(x, y, t)$ is dilated by tB with unit surface speed, where B is any of the following unit-radius 3D summetric convex sets, has a gradient flow governed by the following corresponding PDEs:*

$$B = \text{vert.line} \implies u_t = 1 \tag{20}$$

$$B = \text{horiz.disk} \implies u_t = ||\nabla u|| = \sqrt{u_x^2 + u_y^2} \tag{21}$$

$$B = \text{sphere} \implies u_t = \sqrt{1 + ||\nabla u||^2} \tag{22}$$

with $u(x, y, 0) = u_0(x, y)$.

Proof: Due to lack of space we sketch the main ideas. Write (19) as

$$V(t) = \frac{1}{3}\int_{S_{top}} \vec{S}\cdot\vec{N}d\vec{S} + \text{Area}(R(t)), \quad \dot{V}(t) = \int_{S_{top}} \vec{S}_t\cdot\vec{N}d\vec{S} + \text{const} \tag{23}$$

where S_{top} is the top part of the surface. Over this part we select the optimum surface velocity vector $\vec{S}_t = (x_t, y_t, z_t)$ that maximizes the volume rate of change. Then we exploit the relationships among x, y and the local surface coordinates q_1, q_2 as well as the relation $z(q_1, q_2, t) = u(x, y, t)$ to express u_t as a function of u_x, u_y, which yields the PDE for u. □

So far, we have found a variational interpretation of some well-known multiscale morphological dilations and their corresponding PDEs as area or volume maximization problems. It is straightforward to derive the corresponding multiscale *erosions* and their PDEs by considering the dual problem of area or volume *minimization*. We omit the proofs.

Among the three cases for B, only when B is a vertical line we can also derive the corresponding PDE by using standard variational calculus, as follows.

Proposition 1 *Maximizing the functional* $J[u] = \int\int_R u(x,y,t)dxdy$ *has a gradient flow governed by the PDE* $u_t = 1$.

Proof: By writing $J = \int F$ with $F(u) = u$, the gradient flow will have the general form of (2), i.e. $u_t = [F]_u$. This yields $u_t = 1$, which is the PDE (20). □

In Theorems 1 and 2 we derived the morphological PDEs by maximizing area or volume functionals, either unconstrained if we move in the space of functions (as was the case when B is the vertical line and as explained in Prop.1) or with some geometrical constraints if we move in the space of graphs. Next we interpret our variational results for the multiscale *flat* dilations and erosions as a maximization and minimization, respectively, of the area or volume of the image u but under the constraint that all evolutions u have the same global sup or inf as u_0. This constrained optimization will prove useful for the levelings too.

Theorem 3 *(a) Maximizing the volume functional by keeping invariant the global supremum*

$$\max \int\int_R u\ dxdy \ \ \text{s.t.} \ \ \bigvee u = \bigvee u_0 \tag{24}$$

has a gradient flow governed by the PDE generating flat dilation by disks:

$$u_t = ||\nabla u||, \quad u(x,y,0) = u_0(x,y) \tag{25}$$

Similarly, the dual problem of minimizing the volume functional by keeping invariant the global infimum

$$\min \int\int_R u\ dxdy \ \ \text{s.t.} \ \ \bigwedge u = \bigwedge u_0 \tag{26}$$

has a gradient flow governed by the isotropic flat erosion PDE:

$$u_t = -||\nabla u||, \quad u(x,y,0) = u_0(x,y) \tag{27}$$

(b) For 1D signals $u(x)$*, maximizing (or minimizing) the area functional by keeping invariant the global supremum (or infimum) has a gradient flow governed by the PDE generating flat dilations (or erosions) by intervals* $[-t,t]$*:*

$$\begin{array}{lll} \max \int_R u\ dx \ \ \text{s.t.} \ \bigvee u = \bigvee u_0 & \Longrightarrow & u_t = |u_x| \\ \min \int_R u\ dx \ \text{s.t.} \ \bigwedge u = \bigwedge u_0 & \Longrightarrow & u_t = -|u_x| \end{array} \tag{28}$$

with initial condition $u(x,0) = u_0(x)$.

Proof: Under the sup constraint, the velocity vector for the propagation of the graph of u must have a zero vertical component. Hence, the only directions allowed to propagate the graph of u must be parallel to the image plane. This expansion is done at maximum speed if it corresponds to dilations of the graph (and equivalently of the level sets) of u by horizontal disks in the 2D case and by horizontal line segments in the 1D case. Thus, we have the case of multiscale dilations of the graph of u by horizontal disks or lines for which we use the results of Theorems 1 and 2. Similarly for the erosions. □

4. Variational Approach for Levelings

Here we consider morphological smoothing filters of the reconstruction type. Imagine creating a type of image simplification like a 'cartoon' by starting from a *reference* image $r(x, y)$ consisting of several parts and a *marker* image $u_0(x, y)$ (initial seed) intersecting some of these parts and by evolving u_0 toward r in a monotone way such that all evolutions $u(x, y, t)$, $t \geq 0$, satisfy the following partial ordering, $\forall x, y \in R$

$$t_1 < t_2 \Longrightarrow r(x,y) \preceq_r u(x,y,t_2) \preceq_r u(x,y,t_1) \preceq_r u_0(x,y) \tag{29}$$

The partial order $u \preceq_r f$ means that $r \wedge f \leq r \wedge u$ and $r \vee f \geq r \vee u$. Further, if we partition the following regions R^- and R^+ formed by the zero-crossings of $r - u_0$

$$\begin{aligned} R^- &= \{(x,y) : r(x,y) \geq u_0(x,y)\} = \textstyle\bigsqcup_i R_i^- \\ R^+ &= \{(x,y) : r(x,y) < u_0(x,y)\} = \textstyle\bigsqcup_i R_i^+ \end{aligned} \tag{30}$$

into connected subregions, then the evolution of u is done by maintaining all local maxima and local minima of u_0 inside these subregions R_i^- and R_i^+, respectively:

$$\bigvee_{R_i^-} u = \bigvee_{R_i^-} u_0 \text{ and } \bigwedge_{R_i^+} u = \bigwedge_{R_i^+} u_0, \quad R = (\bigsqcup_i R_i^-) \sqcup (\bigsqcup_i R_i^+) \tag{31}$$

where $\bigsqcup$ denotes disjoint union. Since the order constraint $r \preceq_r u \preceq_r u_0$ implies that $|r-u| \leq |r-u_0|$, the above problem is equivalent to the following constrained minimization

$$\min \iint_R |u - r| dxdy \ \text{ s.t. } \bigvee_{R_i^-} u = \bigvee_{R_i^-} u_0, \ \bigwedge_{R_i^+} u = \bigwedge_{R_i^+} u_0 \tag{32}$$

Theorem 4 *A gradient flow for the optimization problem (32) is given by the following PDE*

$$\begin{aligned} \partial u(x,y,t)/\partial t &= -\mathrm{sgn}(u - r)||\nabla u|| \\ u(x,y,0) &= u_0(x,y) \end{aligned} \tag{33}$$

Proof: By writing the integral $\iint |u - r|$ as

$$\iint_R |u-r| = \sum_{R_i^-} \iint_{R_i^-} (r-u) + \sum_{R_i^+} \iint_{R_i^+} (u-r) \tag{34}$$

we can decompose the global problem (32) into local constraint maximization and minimization problems over the regions R_i^- and R_i^+ respectively. Applying Theorem 3 to these local problems yields local evolutions that act as flat dilations when $u < r$ and as erosions when $u > r$. The PDE (33) has a switch that joins these two actions into a single expression. □

The PDE (33) was introduced in [11] and then studied systematically in [9]. For each t, at pixels (x, y) where $u(x, y, t) < r(x, y)$ it acts as a dilation PDE and hence shifts outwards the surface of $u(x, y, t)$ but does not introduce new local maxima. Wherever $u(x, y, t) > r(x, y)$ the PDE acts as a flat erosion PDE and reverses the direction of propagation. In [9] it was proved that this PDE has a steady-state $u_\infty(x) = \lim_{t\to\infty} u(x, t)$ which is a *leveling* of r with respect to u_0, denoted by $u_\infty = \Lambda(u_0|r)$.

Levelings are nonlinear filters with many interesting scale-space properties [11] and have been used for image pre-segmentation [11, 16]. They were defined geometrically in [10, 11] via the property that if p, q are any two close neighbor pixels then the variation of the leveling between these pixels is bracketed by a larger same-sign variation in the reference image r; i.e., if g is a leveling of r, then

$$g(p) > g(q) \Longrightarrow r(p) \geq g(p) > g(q) \geq r(q) \tag{35}$$

In [9] they were defined algebraically as fixed points of triphase operators $\lambda(f|r)$ that switch among three phases, an expansion, a contraction, and the reference r. Further, the leveling of r w.r.t. $f = u_0$ can be obtained as the limit of iterations of λ:

$$u_\infty = \Lambda(u_0|r) \triangleq \lim_{n\to\infty} \lambda^n(u_0|r) \preceq_r \cdots \lambda(u_0|r) \preceq_r u_0 \tag{36}$$

The simplest choise for λ is $\lambda(f|r) = [r \wedge \delta(f)] \vee \varepsilon(f)$, where δ and ε are dilations and erosions by a small disk, but there are many more sophisticated choises [11, 9]. A numerical scheme proposed in [9] to solve the PDE (33) also involves iterating a discrete algorithm that is essentially a discrete triphase operator whose iteration limit yields a discrete leveling.

Levelings have many interesting scale-space properties [11]. Due to (29) and (35), they preserve the coupling and sense of variation in neighbor image values, which is good for edge preservation. Further, due to (31) the levelings do not create any new regional maxima or minima. In practice, they can reconstruct whole image objects with exact preservation of their boundaries and edges. The reference image plays the role of a global constraint.

5. Conclusions

We have developed a new formulation based on functional extremization to derive the PDEs generating well-known multiscale morphological operators, both of the basic type acting locally on the image like dilations and erosions by compact kernels, as well as of the reconstruction type like the levelings which depend on global constraints. The functionals used were the image volume/area for dilations and the L_1 norm of residuals between the simplified image and the reference for the levelings. Maximization or minimization of these functionals was done subject to some nonlinear constraints. This variational approach to multiscale morphology gives a new insightful interpretation to morphological operators and offers useful links with optimization problems.

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