Morphological Filters—Part II: Their Relations to Median, Order-Statistic, and Stack Filters

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Abstract—**This paper extends the theory of median, order-statistic (OS), and stack filters by using mathematical morphology to analyze them and by relating them to those morphological erosions, dilations, openings, closings, and open-closings that commute with thresholding. The max-min representation of OS filters is introduced by showing that any median or other OS filter is equal to a maximum of erosions (moving local minima) and also to a minimum of dilations (moving local maxima). Thus, OS filters can be computed by a closed formula that involves a max-min on prespecified sets of numbers and no sorting. Stack filters are established as the class of filters that are composed exactly of a finite number of max-min operations.**

The kernels of median, OS, and stack filters are collections of input signals that uniquely represent these filters due to their translation-invariance. The max-min functional definitions of these nonlinear filters is shown to be equivalent to a maximum of erosions by minimal (with respect to a signal ordering) kernel elements, and also to a minimum of dilations by minimal kernel elements of dual filters. The representation of stack filters based on their minimal kernel elements is proven to be equivalent to their representation based on irreducible sum-of-products expressions of Boolean functions.

It is also shown that median filtering (and its iterations) of any signal by convex 1-D windows is bounded below by openings and above by closings; a signal is a root (fixed point) of the median iff it is a root of both an opening and a closing; the open-closing and clos-opening yield median roots in one pass, suppress impulse noise similarly to the median, can discriminate between positive and negative noise impulses, and are computationally less complex than the median. Some similar results are obtained for 2-D median filtering.

I. Introduction

MEDIAN filters and their generalization, order-statistic (in short, OS)[1] filters, are a class of nonlinear and translation-invariant discrete filters that have become popular in digital speech and image processing, and also in statistical or economic time series analysis. These filters are attractive because they are easy to implement and can suppress *impulse noise*[2] while preserving the edges of the signal. This is in contrast to linear filters, which blur edges and only smooth impulses.

Tukey [1] first used the median filter for nonlinear smoothing of data. Median filters were then used in speech smoothing [2], [3] and image enhancement [4]–[7]. Considerable interest and research has been invested in studying properties and fixed points (roots) of median filters [8]–[19]. Roots of the median filter have been used in edge enhancement [1], [6], [7] and image coding [14]. The OS filters were studied in [8], [11], [17], [21], and [22] and used for AM signal detection and image enhancement. Any OS filter can be defined both for binary and for multilevel signals. Moreover, OS filters commute with any monotonic pointwise transformation of the signal amplitude; thresholding at any level in such a monotonic transformation. This important property of OS filters to *commute with thresholding* was investigated independently in [23], [8], [9], [7], [15], [17], [24]–[26]. It was proven for any cascade of min/max operations by Nakagawa and Rosenfeld [23], appeared with examples in Justusson [8], and was stated for the median by Tyan [9], who also notes that a signal is a median root iff each of its thresholded versions (cross sections) is a binary median root. Serra [24] outlined a procedure to prove that the 2-D hexagonal median filter commutes with thresholding. Serra also proved that a signal can be uniquely decomposed into its cross sections and reconstructed from them using a supremum. Fitch *et al.* [15], [17] provided a complete proof, without using morphology, for the combined result that median and OS filters (recursive and nonrecursive) commute with thresholding and hence OS filtering of multilevel signals reduces to a sum of OS filters for binary signals; their result refers to signals with finite extent and a finite number of amplitude levels. Wendt *et al.* [27], [28] defined the stack filters as the class of all filters that are defined via a finite window, commute with thresholding and, hence, are increasing. They made the connection between stack filters and positive Boolean functions [29]–[31], which have a unique minimal expression as Boolean sum of products. Finally, in Maragos and Schafer [25] and Maragos [26], a unified approach was introduced for representing a large class of linear and nonlinear filters (including median and OS filters) as a supremum of erosions or infimum of dilations.

This paper, which reports work from [25] and [26], introduces the use of mathematical morphology, minimal kernel elements, and concepts from Part I [32] to analyze

Manuscript received May 19, 1986; revised March 5, 1987. This work was supported by the Joint Services Electronics Program under Contract DAAG-84-K-0024 at the Georgia Institute of Technology, and in part by the National Science Foundation under Grant CDR-85-00108 at Harvard University.

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IEEE Log Number 8715122.

[1]What we call OS filters in this paper have also been called "ranked-order" operations in [11] and [17]; we use the name OS to keep up with earlier literature [7], [8], [20].

[2]By impulse noise is meant that a signal is corrupted by impulses (spikes), i.e., very large positive or negative values of short duration; probabilistic models of impulse noise can be found in [8].

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median, OS, and stack filters and to relate them with morphological erosions, dilations, openings, and closings. We emphasize at this point that median, OS, and stack filters are related *only* to the function- and set-processing morphological filters that commute with thresholding. The general function-processing morphological filters that involve a nonbinary structuring function do *not* commute with thresholding, and thus are *not* related to median, OS, or stack filters. In Section II we examine some properties of OS filters and provide two alternative simple proofs of the fact that they commute with thresholding. In Sections III and IV we show that: 1) any OS filter can be exactly represented as a maximum of erosions, or as a minimum of dilations; 2) medians and their iterations are bounded below by openings and above by closings; 3) a signal is a median roof iff it is a root of both an opening and a closing; 4) the open–closing and clos–opening give us median roots in a single pass, smooth signals similarly to the median, and have some advantages over the median. Some of the above results are also valid for 2-D signals. In Section V we put result 1) in the unified framework of the theory of minimal elements [26] by introducing the kernel representation of OS filters. Finally, in Section VI we establish that stack filters is the class of all finite min–max and max–min operations, and we relate their representation based on positive Boolean functions to their representation based on minimal elements.

Throughout this paper we use the same notation, terminology, and concepts as in Part I [32].

II. OS Filters for Sets and Functions

We shall deal only with discrete OS filters, i.e., processing sampled signals.[3] Hence, our functions (multilevel signals) will be defined on $\boldsymbol{Z}^m$ (m is any positive integer), and our sets (binary signals) will be subsets of $\boldsymbol{Z}^m$. The functions will generally have their amplitude range in the continuum $\boldsymbol{R}$. Let S be a set of n real numbers, where we allow in S multiple repetitions of the same element. Suppose we sort these n numbers in descending order with respect to their algebraic value; the kth number from this sorted list is called the kth OS of the finite set S, $k = 1, 2, \cdots, n$. If n is odd, for $k = (n+1)/2$ we have the median of S.

Let W be a *window*, which is defined henceforth as a *finite* subset of $\boldsymbol{Z}^m$ with $|W| = n$, where $|\cdot|$ denotes set cardinality. The *kth OS of a function $f(x)$ by W* is the function

$$[\mathrm{OS}^k(f:W)](x) = k\text{th OS of } \{f(y): y \in W_x\}, \quad (1)$$

where $x \in \boldsymbol{Z}^m$, $1 \le k \le n$, and $W_x = \{x + a : a \in W\}$ denotes the set W shifted at location x. The *kth OS filter for functions* by W is a function-processing (FP) filter whose output is the kth OS of the incoming function by W. For $k = (n+1)/2$, whenever n is odd, we have, respectively, the case of the *median* of a function f by W, denoted as med $(f; W)$, and the median filter. If f is a binary function, then its kth OS by W is also a binary function. Thus, FP OS filters are actually function- and set-processing (FSP) filters.

The straightforward way to define OS for sets by a window would be to represent these sets by their *characteristic function* and take the OS of this binary function by this window. An equivalent set-theoretic definition is the following. The *kth OS of a set X by W* is the set

$$\mathrm{OS}^k(X; W) = \{y \in \boldsymbol{Z}^m : |X \cap W_y| \ge k\}, \quad (2)$$

where $k = 1, 2, \cdots, |W|$. Hence, we shift the window W, locate it at y, and *count* the points inside the intersection $X \cap W_y$, where X is the input set. If the number of points is at least k, then the point y belongs to the kth OS of X by W. Note that $|X \cap W_y| \le n$ for all $y \in \boldsymbol{Z}^m$. If n is odd and $k = (n+1)/2$, then the kth OS of X is called the *median* of X by W and denoted as med $(X; W)$. The *kth OS filter for sets by W* is a set-processing (SP) filter whose output is the kth OS of the incoming set by W.

In what follows, the term "OS," except otherwise stated, will always refer to OS of functions or sets by a window. Furthermore, we shall use interchangeably the terms "OS filters for signals" and "OS of signals." OS filters are morphological transformations of signals by sets, because they satisfy all four morphological principles [24]. Moreover, the *window W* is actually a *structuring element* capable of assuming any shape and finite size. It need not be a convex or symmetric set as has been assumed so far by previous researchers. Below we prove some general properties of OS filters.

Property 1: OS filters for sets and functions are increasing.

Proof: Let $X \subseteq Y$. Then $z \in \mathrm{OS}^k(X; W) \Leftrightarrow |X \cap W_z| \ge k$. But, $X \cap W_z \subseteq Y \cap W_z \Rightarrow |X \cap W_z| \le |Y \cap W_z| \Rightarrow z \in \mathrm{OS}^k(Y; W)$. Hence, $\mathrm{OS}^k(X; W) \subseteq \mathrm{OS}^k(Y; W)$. Now, if $f \le g$, then $f(z) \le g(z)$ $\forall z \in W_x$, $\forall x \in \boldsymbol{Z}^m$. Thus, $[\mathrm{OS}^k(f; W)](x) \le [\mathrm{OS}^k(g; W)](x)$, $\forall x$. Q.E.D.

Property 2: OS of functions commute with thresholding. That is, for any function f (of finite or infinite extent) and finite window W, for all $t \in \boldsymbol{R}$ and $k = 1, 2, \cdots, |W|$,

$$X_t[\mathrm{OS}^k(f; W)] = \mathrm{OS}^k[X_t(f); W]. \quad (3)$$

Proof I: Let $g(x) = [\mathrm{OS}^k(f; W)](x)$. Then, $z \in X_t(g) \Leftrightarrow g(z) \ge t \Leftrightarrow |W_z \cap X_t(f)| \ge k \Leftrightarrow z \in \mathrm{OS}^k[X_t(f); W]$. Q.E.D.

Proof II: OS filters are FSP filters that are translation-invariant, increasing, and u.s.c. [26]. Hence, from Theorem 4 of Part I, they commute with thresholding. Q.E.D.

For the median we simply have that $X_t[\mathrm{med}(f; W)] = \mathrm{med}[X_t(f); W]$. Note that Property 2 refers to any signal of finite or infinite extent with a continuous or dicrete amplitude range; in [15] and [17] the commuting with thresholding of OS filters was proved only for the special case of signals with finite extent and a finite num-

[3]For a definition of an analog median filter, see [33].

ber of amplitude levels. The essence of Property 2 is the equivalence between the OS filtering of a function followed by thresholding at level t, on the one hand, and, on the other hand, the thresholding of the function at level t followed by OS filtering of the resulting cross section. That is, both ways should give the same set, i.e., the cross section of the filtered function at level t. In addition, after having obtained all the cross sections of $\mathrm{OS}^k(f; W)$ via OS filtering of the sets $X_t(f)$, we can reconstruct the filtered function by using a supremum (or maximum, for a finite number of amplitude levels). Hence, from Property 2 and Part I (Theorem 1),

$$[\mathrm{OS}^k(f; W)](z) = \sup\left\{t \in \boldsymbol{R}: z \in \mathrm{OS}^k[X_t(f); W]\right\}. \quad (4)$$

In [15], [17], and [28] it is assumed that $f(x)$ has only M (finite) amplitude levels t, and hence the reconstruction (4) can also be done by *summing* the characteristic functions of the sets $\mathrm{OS}^k[X_t(f); W]$ for all M levels t. We can transpose this result to our case, where t varies continuously over $\boldsymbol{R}$, by using *integration*; i.e.,

$$[OS^k(f; W)](z) = \int_{\boldsymbol{R}} \chi_{\mathrm{OS}^k[X_t(f); W]}(z)\, dt \quad (5)$$

where χ_S denotes the *characteristic function* of a set S (i.e., $\chi_S(z) = 1$ for $z \in S$ and $\chi_S(z) = 0$ for $z \notin S$). The reconstructions (4) and (5) are equivalent, and both make use of the fact that $\{\mathrm{OS}^k[X_t(f); W]: t \in \boldsymbol{R}\}$ is a family of decreasing sets as t increases. Consequently, the analysis and implementation of OS filters can be done by focusing only on the case of sets. Clearly, OS of sets are much easier to deal with since their definition involves only *counting* of points instead of *sorting* numbers (as is the case in OS of functions).

Let X^c denote the set complement of X with respect to $\boldsymbol{Z}^m$.

Property 3: OS of sets interact with set complementation as follows. For any set X and finite set W, $\mathrm{OS}^k(X^c; W) = [\mathrm{OS}^{n-k+1}(X; W)]^c$, for $k = 1, 2, \cdots, n = |W|$.

Proof: $z \in \mathrm{OS}^k(X^c; W) \Leftrightarrow |X^c \cap W_z| \geq k \Leftrightarrow |X \cap W_z| < n - k + 1 \Leftrightarrow z \notin \mathrm{OS}^{n-k+1}(X; W) \Leftrightarrow z \in [\mathrm{OS}^{n-k+1}(X; W)]^c$. Q.E.D.

A corollary of the above property, if $|W|$ is odd, is given.

Property 4: Median of sets commutes with complementation; i.e., $\mathrm{med}(X^c; W) = [\mathrm{med}(X; W)]^c$.

III. Relations Between OS and Morphological Filters

By combining the definitions of OS in (1) and (2) with the definitions of erosion and dilation in Part I, we see that the first ($k = 1$) OS of any signal by a window W coincides with its dilation by W. Similarly, the nth OS, where $n = |W|$, is equal to the erosion by W. Hence, it can be shown that, for $k = 1, 2, \cdots, |W|$,

$$X \ominus W^s \subseteq \cdots \subseteq \mathrm{OS}^{k+1}(X; W) \subseteq \mathrm{OS}^k(X; W) \subseteq \cdots \subseteq X \oplus W^s. \quad (6)$$

Thus, the FSP (only) erosions and dilations by sets are special cases of OS filters. Since OS filters commute with thresholding, relation (6) and all subsequent relations involving sets, OS or morphological filters for sets, and set inclusions are also valid for functions too; we only need to replace sets with functions and set inclusion $\subseteq$ with function ordering $\leq$, and vice versa.

We can interpret the kth OS filtering of S by a window W as a cascade of a linear shift-invariant filter with impulse response $h = \chi_W$ followed by the nonlinear pointwise thresholding operation of taking the cross section of $h * \chi_S = \chi_W * \chi_S$ at level $t = k \leq |W|$, where $*$ denotes linear convolution. That is, for all $k = 1, 2, \cdots, |W|$,

$$\mathrm{OS}^k(S; W) = X_{t=k}[\chi_W * \chi_S].$$

This formula allows us to implement OS filters (including SP erosion and dilation) in terms of linear convolutions; obviously, this is a *serial* implementation.

Next we show how any OS filter can be expressed as a maximum of erosions or minimum of dilations. Let $\binom{n}{k} = n!/k!(n-k)!$ denote the number of combinations of n items grouped k at a time, where $0! = 1$.

Theorem 1: For any function f and any finite set W, the kth OS of f by W, $k = 1, 2, \cdots, n = |W|$, is equal to the pointwise maximum of the moving local minima of f inside all $\binom{n}{k}$ windows equal to the subsets of W containing exactly k points, and it is also equal to the minimum of the moving local maxima of f inside all the subsets of W containing exactly $n - k + 1$ points. As a special case, the median of f by W is equal to the maximum of minima (and also the minimum of maxima) of f inside all subsets of W containing exactly $(n + 1)/2$ points, where n is odd.

Proof: Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be the n ordered values of f inside W shifted to any location z. Let S_i, $i = 1, 2, \cdots, \binom{n}{k}$, be the sets of values of f on each of all the different subsets of W_z containing exactly k points. Since $\{a_1, \cdots, a_{k-1}, a_k\}$ is one of the S_i's, then a_k is one of the minima of f on the S_i's. Every other set S_i will have at least one element from the set $\{a_{k+1}, a_{k+2}, \cdots, a_n\}$ and, hence, it will have a minimum $\leq a_k$. Thus, the maximum of all these minima is equal to $a_k = [\mathrm{OS}^k(f; W)](z)$. Likewise, $T_1 = \{a_k, a_{k+1}, \cdots, a_n\}$ is one of the sets T_m, $m = 1, 2, \cdots, \binom{n}{n-k+1}$, of the values of f on the subsets of W_z containing $n - k + 1$ points. Clearly, $a_k = \max(T_1)$. Every other set $T_m \neq T_1$ will have at least one element from the set $\{a_1, a_2, \cdots, a_{k-1}\}$ and, hence, it will have a maximum $\geq a_k$. Thus, the minimum of all these maxima is again equal to a_k. For the case of the median, n is odd and $k = (n + 1)/2 \Rightarrow k = n - k + 1$. Hence, the subsets of W with k points are equal to the subsets with $n - k + 1$ points. Thus, the minima and maxima refer to the same subsets of W. Therefore, the median can be expressed both as a maxi-

mum of minima and as a minimum of maxima on all the subsets of W containing exactly $(n+1)/2$ points.

Q.E.D.

From Theorem 1 the following theorem immediately results.

Theorem 2: The kth OS of sets (respectively, functions) by a window W, $k = 1, \cdots, n = |W|$, is equal to the union (respectively, maximum) of erosions by all the subsets of W containing k points. It is also equal to the intersection (respectively, minimum) of dilations by all the subsets of W containing $n - k + 1$ points. That is, for any set $S \subseteq \mathbf{Z}^m$,

$$\mathrm{OS}^k(S; W) = \bigcup_{\substack{P \subseteq W \\ |P| = k}} S \ominus P^s = \bigcap_{\substack{Q \subseteq W \\ |Q| = n-k+1}} S \oplus Q^s. \tag{7}$$

For any function $f(x)$, $x \in \mathbf{Z}^m$, and P, Q as in (7), we have

$$[\mathrm{OS}^k(f; W)](x) = \max_P \{(f \ominus P^s)(x)\} = \min_Q \{(f \oplus Q^s)(x)\}. \tag{8}$$

If n is odd, for $k = (n+1)/2$ we have the special case of the median:

$$[\mathrm{med}(f; W)](x) = \max_{\substack{B \subseteq W \\ |B| = (n+1)/2}} \{(f \ominus B^s)(x)\} = \min_{\substack{B \subseteq W \\ |B| = (n+1)/2}} \{(f \oplus B^s)(x)\}. \tag{9}$$

The median of S by W is given from (7) by setting $k = (n+1)/2$.

Proof: Equation (8) results from Theorem 1 and the fact that the local minimum (maximum) filter with respect to a moving window A is equal to the erosion (dilation) by A. Equation (7) results from (8) by setting $S = X_t(f)$ for some t and an arbitrary function f, because $X_t[\mathrm{OS}^k(f; W)] = \mathrm{OS}^k(S; W)$, $X_t[\max_P\{f \ominus P^s(x)\}] = \cup_P X_t(f \ominus P^s) = \cup_P S \ominus P^s$, and $X_t[\min_Q\{f \oplus Q^s(x)\}] = \cap_Q X_t(f \oplus Q^s) = \cap_Q S \oplus Q^s$. If n is odd and $k = (n+1)/2 = n - k + 1$ we get (9) from (8) or (7).

Q.E.D.

IV. Medians, Openings, Closings

So far, our discussion has been general and referred to every OS by an arbitrary window W. In this section we discuss only the case of *median* filtering by *convex* windows, because this constraint enables us to find some interesting properties between such median filters and openings/closings. An intuitive idea about such properties can be obtained from Fig. 1. Fig. 1(a) shows a function f representing a 256 × 256-pixel graytone image corrupted by salt-and-pepper noise. In Fig. 1(b) the opening f_B of f by a 2 × 2-pixel square convex set B cuts down the peaks of f and hence suppresses the positive noise spikes ("salt" noise). In Fig. 1(c) the *open–closing* $(f_B)^B$ fills up the valleys of f_B and hence suppresses the negative noise spikes ("pepper" noise). Comparing Fig. 1(c) and (d) indicates that a median filtering of f by a 3 × 3-pixel convex square window W behaves similarly to the open–closing by B, but the latter is computationally less complex than the median. In addition, the open–closing can decompose the noise suppression task into two parts; i.e., opening suppresses the positive noise impulses, the closing suppresses the negative noise impulses, but the median cannot discriminate between them. Qualitatively, the median behaves like a combined opening and closing by a set of size about half the size of the median window. Next we formalize our discussion.

Fig. 1. (a) A 256 × 256-pixel (8-bit/pixel) graytone image f corrupted with *salt-and-pepper* noise; SNR = 15.1 dB. (Probability of occurrence of noisy samples is 0.1.) (b) Opening f_B of f by a 2 × 2-pixel square set B; SNR = 19.5 dB. (c) Open–closing $(f_B)^B$; SNR = 25.8 dB. (d) Median of f by a 3 × 3-pixel window; SNR = 29.1 dB. (The SNR's were computed by $20 \log_{10}(255/e_{\mathrm{rms}})$, where e_{rms} was the rms-value of the difference between the original and the noisy or restored images.)

A. Medians by 1-D Convex Windows

Assume in this section that the window for median filtering is a *convex symmetric* (with respect to the origin) set W and that the structuring element for openings and closings is a convex set B, where $W, B \subseteq \mathbf{Z}$,

$$|W| = 2n + 1, \quad |B| = n + 1, \quad n \in \mathbf{Z}_+, \tag{10}$$

and $\mathbf{Z}_+$ is the set of positive integers. The set B does not have to be symmetric or contain the origin, because the opening by B is equal to the opening by any translation B_y of B. That is,

$$X_B = \bigcup_{B_a \subseteq X} B_a \overset{a=y+b}{=} \bigcup_{(B_y)_b \subseteq X} (B_y)_b = X_{(B_y)}$$

for any $y \in \mathbf{Z}$; likewise for the closing by B.

The input signals to the examined filters will be sets or functions, and, if not otherwise stated, of finite or infinite extent. In proving the theoretical results of this section we assume, for simplicity, that the signals are 1-D. How-

ever, the obtained results are also valid for *multi-D* input signals. This is true because we can "slice" (intersect) a multi-D signal by all 1-D discrete lines $\boldsymbol{Z}$ parallel to the line containing the 1-D median window W and essentially reduce the multi-D filtering into 1-D filtering of each 1-D slice.

A root (or *fixed point*) of a filter $\psi(\cdot)$ is any signal f such that $\psi(f) = f$. If ψ is FSP and commutes with thresholding, then a function f is a function-root of ψ iff all the cross sections of f are set-roots of ψ. Let $\text{med}^{\circ k}(\cdot; W)$, $k \in \boldsymbol{Z}_+$, denote the kth *iteration* of the median filter by W, where $\text{med}^{\circ(k+1)}(f; W) = \text{med}[\text{med}^{\circ k}(f; W); W]$ and $\text{med}^{\circ 1}(f; W) = \text{med}(f; W)$. If f has a finite extent, then iterating the median by W on f will yield a *median root*, which we denote by $\text{med}^{\circ\infty}(f; W)$; actually, this root will be obtained after only a finite number of iterations [10], [18].

The ordering relations (6) show that the median of a signal by a window W is bounded below by its erosion and above by its dilation by W. Below we give tighter bounds for the median and its iterations, and provide a sufficient condition to find a median root.

Theorem 3: The median (and any of its iterations) of any set X or function f by W is bounded below by the opening and above by the closing of the signal by B. Further, if the signal is a root of the opening and closing by B, it is a root of the median by W.

a) $X_B \subseteq \text{med}^{\circ k}(X; W) \subseteq X^B$, and $f_B \leq \text{med}^{\circ k}(f; W) \leq f^B$, $\forall k \in \boldsymbol{Z}_+$.

b) $X = X_B = X^B \Rightarrow X = \text{med}(X; W)$, and $f = f_B = f^B \Rightarrow f = \text{med}(f; W)$.

Proof: Let $k = 1$. Then $z \in X_B \Rightarrow$ there is y such that $z \in B_y \subseteq X$. From (10) and since $W = W^s$, $|W_z \cap B_y| \geq n + 1 \Rightarrow |W_z \cap X| \geq n + 1 \Rightarrow z \in \text{med}(X; W)$. Thus, $X_B \subseteq \text{med}(X; W)$. Now $(X^c)_B \subseteq \text{med}(X^c; W) \Rightarrow (X^B)^c \subseteq [\text{med}(X; W)]^c \Rightarrow \text{med}(X; W) \subseteq X^B$. The median is increasing filter. Hence, $\text{med}(X_B; W) \subseteq \text{med}^{\circ 2}(X; W) \subseteq \text{med}(X^B; W)$. But $X_B = (X_B)_B \subseteq \text{med}(X_B; W)$ and $\text{med}(X^B; W) \subseteq (X^B)^B = X^B$. Thus, $X_B \subseteq \text{med}^{\circ 2}(X; W) \subseteq X^B$, and by repeating the same procedure on the latter result we obtain $X_B \subseteq \text{med}^{\circ k}(X; W) \subseteq X^B$, $\forall k \in \boldsymbol{Z}_+$. The FSP filter $\text{med}^{\circ k}(\cdot; W)$ commutes with thresholding because it is a cascade of k medians. Thus, by setting in the previous proof $X = X_t(f)$, it follows that $\forall t \in \boldsymbol{R}$, $[X_t(f)]_B \subseteq \text{med}^{\circ k}[X_t(f); W] \subseteq [X_t(f)]^B \Leftrightarrow X_t(f_B) \subseteq X_t[\text{med}^{\circ k}(f; W)] \subseteq X_t(f^B) \Leftrightarrow f_B \leq \text{med}^{\circ k}(f; W) \leq f^B$.

Theorem 3b) is a simple corollary of a) since $X = X_B \subseteq \text{med}(X; W) \subseteq X^B = X \Rightarrow X = \text{med}(X; W)$. Likewise for functions. Q.E.D.

Note that we take the median by W, but the opening and closing by B. From (10), the set W is a fixed point of both the opening and closing by B. Hence, from [24], $X_W \subseteq X_B$ and $X^B \subseteq X^W$. Therefore, if we take the opening and closing by W instead of B, we will bound the median with looser bounds.

By restricting the signal to be of finite extent, we can find a necessary and sufficient condition relating the median roots to the roots of the opening and closing. Gallagher and Wise [10] proved that a multilevel signal of finite length is a median root by a window of $2n + 1$ points iff it consists of edges (monotonic regions) and constant neighborhoods of at least $n + 1$ consecutive points. Using their method of proof, we now prove a similar theorem for sets (our approach differs in the way we handle the boundary conditions).

Theorem 4: A finite set X is a root of the median by W, $|W| = 2n + 1$, iff it consists of convex subsets of length at least $n + 1$ points and these subsets are separated from each other by convex subsets of X^c of at least $n + 1$ points.

Proof (Sufficiency): Both X and X^c consist of convex subsets of length $\geq n + 1 = |B|$. Hence, $X = X_B$, and $X^c = (X^c)_B \Leftrightarrow X = X^B$; thus, $X = \text{med}(X; W)$ due to Theorem 3.

Necessity: Let $X = \text{med}(X; W)$, as shown below for a set X with $W = \{-2, -1, 0, 1, 2\}$:

$$X = \cdots 0000\overset{a}{\underset{\downarrow}{1}}11\ 1111\overset{b}{\underset{\downarrow}{1}}000\ 0111000 \cdots . \tag{11}$$

$$\underbrace{}_{W_a \rightarrow} \qquad \underbrace{}_{W_b \rightarrow}$$

Slide the window W of $2n + 1$ points from left to right (in (11), $n = 2$). At the first point a of X, the left part of the window W_a contains n points of X^c. Thus, the right part of the window W_a must contain n points of X so that $a \in \text{med}(X; W)$. Thus, adjacent to the point a from the right there is a convex subset of X of length $k \geq n + 1$ points (in (11), $k = 9$) including point a. All these k points will remain after median filtering. Moving the window W from left to right, after this k-point subset of X, we will encounter points of X^c; call b the first such point. The window W_b contains from the left n points of X, and, hence, it must contain from the right n points of X^c, so that $b \notin \text{med}(X; W)$. Thus, the point b must see from the right a convex subset of X^c of length $j \geq n + 1$ points (in (11), $j = 4$) including point b. All these j background points will remain unchanged after median filtering. Continuing to the right we may encounter another point of X; by repeating the above process, we complete the proof. Q.E.D.

In the above theorem and in all our analysis concerning OS filtering, we did not assume, as in [10], that the finite extent signal was extended by appending values at the border points. We simply let the operation of set complementation take care of the border points. That is, every point outside the finite extent of the set-signal belongs to the background. Based on Theorem 4 we can now relate the roots of the median with those of the opening and closing, as follows.

Theorem 5: A set X or function f of finite extent is a root of the median by W iff it is a root of both the opening and the closing by B. That is, $X = \text{med}(X; W) \Leftrightarrow X = X_B = X^B$, and $f = \text{med}(f; W) \Leftrightarrow f = f_B = f^B$.

Proof:

1) For Sets: Sufficiency results from Theorem 3b).

Necessity: Let $X = \text{med}(X; W)$. Then, from Theorem 4, both X and X^c consist of convex subsets of length $\geq n + 1 = |B|$. Hence, $X = X_B$, and $X^c = (X^c)_B \Leftrightarrow X = X^B$.

2) For Functions: Using the links between sets and functions and the fact that median, opening, and closing commute with thresholding, we have: $f = \text{med}(f; W) \Leftrightarrow X_t(f) = Y = \text{med}(Y; W)\ \forall t \in \boldsymbol{R} \Leftrightarrow Y = Y_B = Y^B \Leftrightarrow f = f_B = f^B$. Q.E.D.

For infinite signals, Theorem 3b) is still true, whereas Theorem 5 is not always true, as the following counterexample shows. Consider the *infinite* 1-D set X below and let $B = \{-1, 0, 1\}$. Then,

$$X = \cdots 0\,1\,0\,1\,0\,1\,0\,1 \cdots \Rightarrow X_B$$
$$= \cdots 0\,0\,0\,0\,0\,0\,0\,0 \cdots$$

where $0 \in X^c$ and $1 \in X$. Then, if $W = \{-2, -1, 0, 1, 2\}$, $X = \text{med}(X; W)$, but $X \neq X_B$.

One implication of Theorem 5 is the idea that instead of iterating a median filter many times to obtain a median root, one could alternatively obtain a signal that is a root of the opening and closing; this latter signal would then be a median root. In contrast to the median filter, to obtain a root of the opening or closing we need not iterate the opening or closing, respectively, because both operations are idempotent. That is, the opening or closing of a signal is itself a root of the opening or closing filter, respectively.

A morphological filter that yields roots of *both* the opening and closing, and hence, of the median, is the *open–closing* or *clos–opening*. The open–closing (opening followed by closing by the same structuring element) of X by B is equal to $(X_B)^B$. The clos–opening (closing followed by opening) of X by B is equal to $(X^B)_B$. Likewise for functions. Before we prove the above assertion we need Theorem 6.

Theorem 6: For any set X or function f of finite extent, the root of the median by W is bounded below by the open–closing and above by the clos–opening by B. That is, $(X_B)^B \subseteq \text{med}^{\circ\infty}(X; W) \subseteq (X^B)_B$, and $(f_B)^B \leq \text{med}^{\circ\infty}(f; W) \leq (f^B)_B$.

Proof: After a finite number, say k, of iterations of the median, we obtain the median root $Y = \text{med}^{\circ\infty}(X; W) = \text{med}^{\circ k}(X; W)$. Then, from Theorem 3, $X_B \subseteq Y \subseteq X^B$. Since Y is a median root, from Theorem 5, $Y = Y_B = Y^B$. Thus, $(X_B)^B \subseteq Y^B = Y = Y_B \subseteq (X^B)_B$, because opening and closing are increasing. Similarly for functions (by considering their cross sections since all the examined filters commute with thresholding). Q.E.D.

So far we have seen that the median is bounded by the opening and closing and that the median root of a finite signal is bounded by the open–closing and the clos–opening. Below we prove that these two latter morphological filters are median roots by themselves.

Theorem 7: The open–closing and clos–opening by B of any finite extent function f or set X are roots of the median by W. That is, $(f_B)^B = \text{med}[(f_B)^B; W]$, and $(f^B)_B = \text{med}[(f^B)_B; W]$; likewise, if a set X replaces f.

Proof: From Theorem 6, $(f_B)^B = [(f_B)_B]^B \leq \text{med}^{\circ\infty}(f_B; W) \leq [(f_B)^B]_B \leq (f_B)^B$. Hence, $(f_B)^B = [(f_B)^B]_B$ is a root of the opening and, obviously, of the closing by B. Thus, from Theorem 3b), $(f_B)^B$ is a root of the median by W. Similarly, $(f^B)_B$ is equal to its opening and closing by B and, hence, a median root by W. Likewise for sets, if we replace in the previous proof f with X and $\leq$ with $\subseteq$. Q.E.D.

Fig. 2(a) shows a finite 1-D multilevel signal f of 256 samples representing a graytone image intensity profile. Fig. 2(b), (c), and (d) shows, respectively, the open–closing and clos–opening of f by B, and the median root of f by W, where $|B| = 3$ and $|W| = 5$. The median root was obtained by iterating the median four times. The bounds of Theorem 6 are satisfied, but the difference between the median root and the open–closing or clos–opening is very small. For a 1-D signal of L samples, the maximum number of iterations of the median filter needed to obtain the median root (with respect to a $2n + 1$-point window) is equal to $3(L - 2)/(2n + 4)$ if we append samples at the ends [18], and $3L/(2n + 4)$ if we do not append samples (in Fig. 2, $L = 256$, $n = 2$). Further, for a 256-sample signal, 28 iterations of the median are needed [16], at most, to obtain a signal which is a median root with a confidence of 95 percent. However, Theorems 6 and 7 tell us that the open–closing and clos–opening yield in *one pass* median roots, which bound from below and above, respectively (and, as Fig. 2 shows, lie close to) the median root obtained by iterating the median. As an aside, both open–closing and clos–opening are *idempotent* operations [26], [32], and, hence, their output stabilizes in a single pass. In addition, if we view Fig. 2(a) as a signal corrupted with impulse noise, and Fig. 2(b), (c), and (d) as its smoothed versions, then clearly the open–closing (or clos–opening) behave very similarly to the median. Moreover, as in Fig. 1, the open–closing can selectively eliminate positive or negative noise impulses, whereas the median root cannot discriminate them. In [34], the statistical properties of 1-D and 2-D open–closings (by a combination of 4 oriented $n + 1$-point 1-D sets) and median filters (by the same combination of $2n + 1$-point windows) were compared; it was found that these median filters offer more noise suppression, whereas open–closings appear to have superior syntactical performance.

It may be of interest to compare the computational complexity involved to obtain the open–closing (or, equivalently, the clos–opening) and a median root. Recall that $|W| = 2n + 1$ and $|B| = n + 1$. Then the following three quantities are involved in measuring the computational complexity (per output sample): the number of *passes* P; the number of *exchanges* E; and the number of *comparisons* C. For the open–closing by B (two local min and two local max), we have $P = 4$, $E = 0$, and $C = 4n$. For a single median filtering operation by W, using the

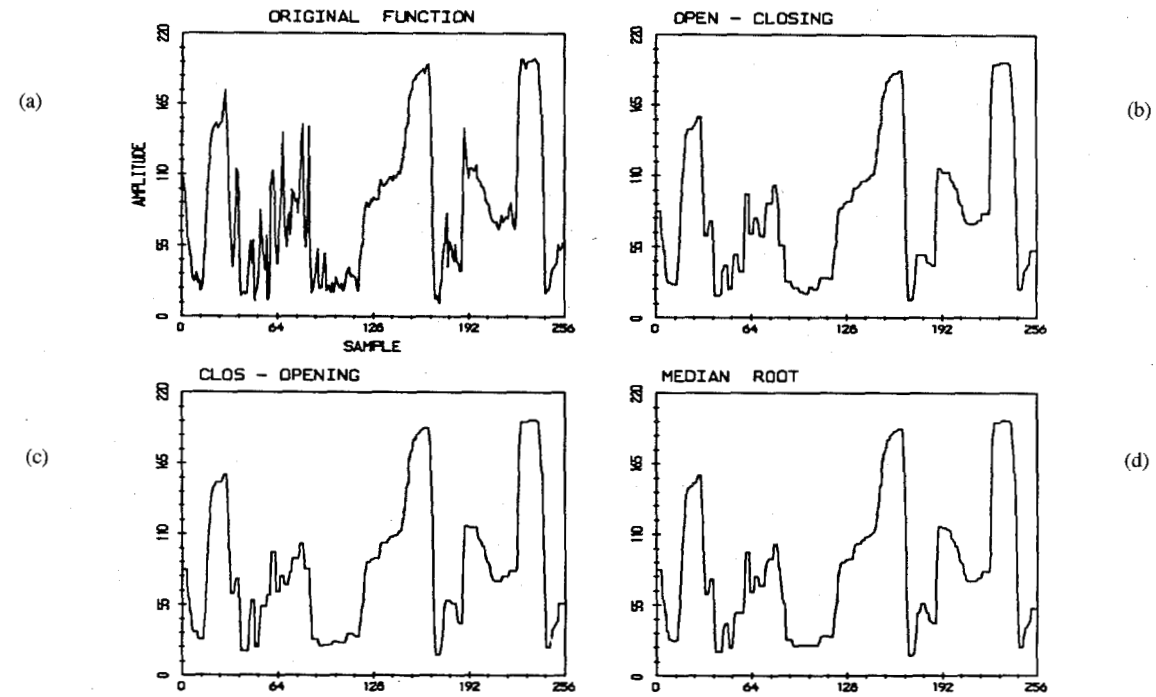

Fig. 2. (a) Original 1-D function f. (b) Its open-closing $(f_B)^B$ by $B = \{-1, 0, 1\}$. (c) Its clos-opening $(f^B)_B$. (d) Its median root $\text{med}^{\circ\infty}(f; W)$ by $W = \{-2, -1, 0, 1, 2\}$.

bubble-sort [35] algorithm for sorting, we have $1 \le P \le 2n + 1$, $0 \le E \le (2n^2 + n)$, and $2n \le C \le (2n^2 + n)$. In addition, the *average number* of these three quantities required for the median (i.e., for the sorting) is approximately [35] of the following order: $O(M)$ for P_{ave}; $O(M^2)$ for E_{ave}; and $O(M^2)$ for C_{ave}; where $M = 2n + 1$. Of course, there are other faster algorithms for sorting [35] or specifically for the median (see [6] for references), but these faster algorithms usually come with an increase in sophistication. Thus, the open–closing (or clos–opening) requires a comparable and, in many cases, smaller computational complexity than a single median. In addition, the iterations of the median needed to obtain median roots compared to the single pass needed for the open–closing (or clos–opening) make the latter more appealing.

In summary, the following orderings and bounds have been established between openings, closings, medians, and median roots:

$$X_B \subseteq \text{med}(X_B; W) \subseteq (X_B)^B \subseteq \text{med}^{\circ\infty}(X; W) \subseteq (X^B)_B \subseteq \text{med}(X^B; W) \subseteq X^B. \tag{12}$$

Similar results hold for functions too:

$$f_B \le \text{med}(f_B; W) \le (f_B)^B \le \text{med}^{\circ\infty}(f; W) \le (f^B)_B \le \text{med}(f^B; W) \le f^B. \tag{13}$$

B. Medians by 2-D Windows

Some of the results concerning medians by 1-D convex windows also apply for median filtering of multi-D discrete sequences by certain 2-D windows. Two such windows shaped like a 7-point hexagon are shown in Fig. 3; the window H is for *hexagonally sampled* 2-D signals, and the window R for *rectangularly sampled* 2-D signals.

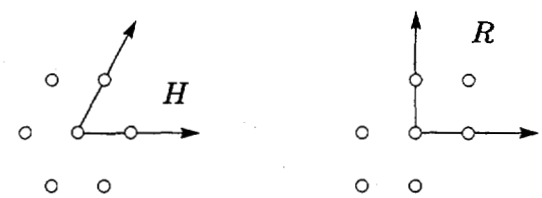

Fig. 3. 2-D median filtering windows: H for hexagonal grids and R for rectangular grids.

Consider any subset X of a plane $\mathbf{Z}^2$ whose points are arranged as on a hexagonal grid. Then the set H has the property that

$$z \in X_H \in |H_z \cap X| \ge 4, \tag{14}$$

since $z \in X_H$ iff there is point y such that $z \in H_y \subseteq X$, and the latter implies $|H_z \cap X| \ge 4$ due to the geometry of H. Based on (14) we have Theorem 8.

Theorem 8: For any set X or function f defined on a hexagonal grid:

a) $X_H \subseteq \text{med}^{\circ k}(X; H) \subseteq X^H$, and $f_H \le \text{med}^{\circ k}(f; H) \le f^H$, $\forall k \in \mathbf{Z}_+$.
b) $X = X_H = X^H \Rightarrow X = \text{med}(X; H)$, and $f = f_H = f^H \Rightarrow f = \text{med}(f; H)$.

Proof: a) From (14), $z \in X_H \Rightarrow x \in \text{med}(X; H)$, and hence $X_H \subseteq \text{med}(X; H)$. Then $(X^c)_H \subseteq \text{med}(X^c; H) \Leftrightarrow (X^H)^c \subseteq [\text{med}(X; H)]^c \Leftrightarrow \text{med}(X; H) \subseteq X^H$. By following the same procedure as in the proof of Theorem 3a), we can complete the proof of a) both for sets and functions. Finally, b) is a simple corollary of a). Q.E.D.

Theorem 8b) is true both for finite and infinite signals;

the converse, even for finite signals, is not generally true. We prove this through a counterexample shown in Fig. 4, where we see a finite set Y for which $Y = \text{med}(Y; H)$, but $Y \neq Y^H$. Further, Theorem 7 is not valid for 2-D filtering by H because neither the open–closing nor the clos–opening by H are always roots of both the opening and closing by H. We prove this through a counterexample. If Y is the set of Fig. 4, $Y^H = (Y_H)^H \neq [(Y_H)^H]_H = Y$; thus $(Y_H)^H$ is not a root of the opening and not a median root by H.

Finally, Theorem 8 also applies to median filtering of rectangularly sampled functions f or sets X by the window R of Fig. 3. That is, $f_R \leq \text{med}^{\circ k}(f; R) \leq f^R$, and $f = f_R = f^R \Rightarrow f = \text{med}(f; R)$; the proof proceeds exactly as in Theorem 8, and exploits the fact that $z \in X_R \Rightarrow |R_z \cap X| \geq 4$, due to the geometry of R.

V. Representation of OS Filters by Kernels and Minimal Elements

In this section we present the relations between morphological and OS filters under the unified framework of the theory presented in [25] and [26]. Consider the SP filter $\Psi(X) = \text{OS}^k(X; W)$, where $k = 1, 2, \cdots, |W| = n$ and $X \subseteq \mathbf{Z}^m$. Then the *dual* SP filter of Ψ with respect to complementation, defined by $\Psi^d(X) = [(X^c)]^c$, $X \subseteq \mathbf{Z}^m$, is equal to the $(n - k + 1)$th OS filter by W. For instance, if Ψ is the set erosion, then Ψ^d is the set dilation by W, and vice versa. If n is odd and $k = (n + 1)/2$, then both Ψ and Ψ^d coincide with the SP median by W. The filter Ψ is *translation-invariant* (in short, *TI*) and is defined on $\mathcal{P}(\mathbf{Z}^m)$ (the class of all subsets of $\mathbf{Z}^m$), which is a class of sets closed under set translation. The *kernel* of Ψ is defined generally by $\mathcal{K}(\Psi) = \{X \subseteq \mathbf{Z}^m : \mathbf{0} \in \Psi(X)\}$, where $\mathbf{0}$ is the zero vector of $\mathbf{Z}^m$. Thus, $\mathcal{K}(\Psi)$ is a collection of input sets that can uniquely characterize and reconstruct Ψ by using translations [36]. That is, $\Psi(X) = \{a \in \mathbf{Z}^m : X_{-a} \in \mathcal{K}(\Psi)\}$, for each X. From (2),

$$\mathcal{K}(\Psi) = \{X \subseteq \mathbf{Z}^m : |X \cap W| \geq k\}. \tag{15}$$

In [25] and [26], we extended the kernel representation to FP filters that are TI (see Table II of Part I) and are defined on a class $\mathcal{F}$ of functions closed under function translation. The FSP filter $\psi(f) = \text{OS}^k(f; W)$ is such a case with $\mathcal{F}$ being, for example, the class of all real-valued functions f defined on $\mathbf{Z}^m$. The kernel of ψ, defined generally by $\mathcal{K}(\psi) = \{f \in \mathcal{F} : [\psi(f)](\mathbf{0}) \geq 0\}$, is a collection of input functions that can uniquely characterize and reconstruct ψ by using translations and supremum [26]. Since $X_t[\psi(f)] = \Psi[X_t(f)]\ \forall t \in \mathbf{R}, f \in \mathcal{K}(\psi) \Leftrightarrow X_{t=0}(f) \in \mathcal{K}(\Psi)$.

Since Ψ is TI and *increasing*, it can be realized as the union of SP erosions by all its kernel elements [36]. Similarly, since ψ is TI, increasing, and commutes with thresholding, it can be realized exactly as the pointwise supremum of FSP erosions by all the kernel elements of Ψ [26]. However, these realizations, except for their theoretical interest, are impractical because they require an infinite number of kernel elements. This is why we introduced the concept of the basis $\mathcal{B}(\Psi)$ of Ψ, which is defined as the set of *minimal elements* of $\mathcal{K}(\Psi)$. [The system $(\mathcal{K}(\Psi), \subseteq)$ is a partially ordered set; an element $M \in \mathcal{K}(\Psi)$ is minimal iff, for each $A \in \mathcal{K}(\Psi)$, $A \subseteq M \Rightarrow A = M$.] These minimal elements exist in $\mathcal{K}(\Psi)$ if Ψ is TI, increasing, and upper semicontinuous (u.s.c.) [26]. Next we provide a representation theorem for filters like Ψ based only on their basis.

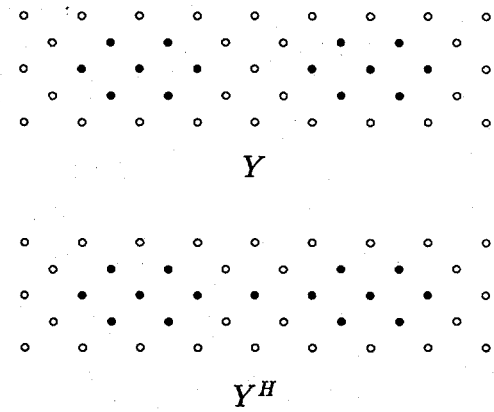

Fig. 4. Counterexample for relations between 2-D medians and openings–closings. ($\bullet \in Y$, $\circ \in Y^c$.)

Theorem 9 (Maragos [26, pp. 136–137]):

a) Let $\Phi: \mathcal{S} \to \mathcal{P}(\mathbf{Z}^m)$ be a TI, increasing, and u.s.c. SP discrete filter, where $\mathcal{S} \subseteq \mathcal{P}(\mathbf{Z}^m)$ is closed under translation and infinite intersection. Then Φ is exactly represented as the union of erosions by its basis sets. If its dual Φ^d is u.s.c., Φ can also be represented as the intersection of dilations by the basis sets of Φ^d.

b) Let ϕ be a TI discrete FSP filter commuting with thresholding, and Φ be its respective SP filter. Then ϕ is exactly represented as the pointwise supremum of FSP erosions by the basis sets of Φ. If the dual of Φ is u.s.c., ϕ can also be represented as the infimum of FSP dilations by the basis sets of Φ^d.

Note that in Theorem 9b), Φ and Φ^d are TI because ϕ is TI. Further, Φ is increasing and u.s.c. because ϕ commutes with thresholding (see Section II-D of Part I); since Φ is TI and increasing, Φ^d is TI and increasing too. Theorem 9 applies to median and OS filters as follows.

Theorem 10:

a) Let the m-D SP filter Ψ be the kth OS of sets by $W \subseteq \mathbf{Z}^m$, where $k = 1, 2, \cdots, |W| = n$; its dual Ψ^d is the $(n - k + 1)$th OS filter by W. The minimal kernel elements of Ψ are all the $\binom{n}{k}$ subsets P of W with $|P| = k$, and the minimal kernel elements of Ψ^d are all the $\binom{n}{n-k+1}$ subsets Q of W with $|Q| = n - k + 1$. The filter Ψ is equal to the union of set erosions by all P's and the intersection of dilations by all Q's.

b) If the m-D FSP filter ψ is the kth OS of functions by W, then ψ is equal to the pointwise maximum of function erosions by all the P's and to the minimum of dilations by all the Q's. [See also (7)–(9).]

Proof:

a) From (15), if $P \subseteq W$ with $|P| = k$, then $P \in \mathcal{K}(\Psi)$.

Now, if there is $G \in \mathcal{K}(\Psi)$ such that $G \subseteq P$, $k \leq |G \cap W| \leq |G| \leq |P| = k \Rightarrow G = P$. Hence, P is a minimal element in $(\mathcal{K}(\Psi), \subseteq)$. For any other $X \in \mathcal{K}(\Psi)$, $X \supseteq F = X \cap W$ with $F \subseteq W$ and $|F| \geq k$. Then F, and thus X, contains a subset P of W with $|P| = k$. Hence, the subsets P of W with $|P| = k$ are the only minimal elements in $\mathcal{K}(\Psi)$. Since the dual filter Ψ^d is the $(n - k + 1)$th OS by W, the minimal elements of $\mathcal{K}(\Psi^d)$ are all the $Q \subseteq W$ such that $|Q| = n - k + 1$. The filters Ψ and Ψ^d, defined on $\mathcal{P}(\boldsymbol{Z}^m)$, are TI, increasing, and u.s.c. [26]. Hence, from Theorem 9, Ψ is the union of erosions by all P's and the intersection of dilations by all Q's.

b) For the FSP filter ψ, if the input function is f, and $Y = X_t(f)$ for any $t \in \boldsymbol{R}$, then $\Psi(Y)$ is the union of all $Y \ominus P^s$ and the intersection of all $Y \oplus Q^s$. Since $\Psi(Y) = X_t[\psi(f)]$, $\psi(f)$ is the maximum of all $f \ominus P^s$ and the minimum of all $f \oplus Q^s$. Q.E.D.

Basically, in Theorem 10 we proved the same result as in Theorem 2 but without using Theorem 1; we used instead the minimal kernel elements of OS filters. Clearly, the advantage of the minimal elements approach is that it unifies the representation of OS filters as well as of many other filters [26], e.g., linear filters (see Part I), in terms of morphological filters. The basis elements of openings and closings are given by Theorem 11.

Theorem 11 (Maragos [26, pp. 141–143]: Let $A \subseteq \boldsymbol{Z}^m$ with $|A| = n$. Then:

a) the basis of the SP opening filter $\Psi(X) = X_A$, $X \subseteq \boldsymbol{Z}^m$, consists of the n sets A_{-a} where $a \in A$; and

b) the basis of the SP closing filter $\Psi^d(X) = X^A$ consists of all minimal subsets M of $A \oplus A^s$ such that $\mathbf{0} \in M^A$. If A is 1-D and convex, then the basis of Ψ^d consists of the set $\{0\}$ and the $n(n-1)/2$ sets $\{a, b\} \subseteq A \oplus A^s$ such that $0 \in \{a, b\}^A$.

Next we provide some examples to clarify the basis representation of median, opening, and closing filters.

Example 1 (Median): Consider first the 1-D SP median $\Psi(X) = \text{med}(X; W)$, where $X \subseteq \boldsymbol{Z}$ and $W = \{-1, 0, 1\}$. The kernel of Ψ is $\mathcal{K}(\Psi) = \{X : |X \cap W| \geq 2\}$. The kernel elements have the form $\{\cdots, -1, 0, \cdots\}$, or $\{\cdots, -1, 1, \cdots\}$, or $\{\cdots, 0, 1, \cdots\}$. Clearly, there is an infinite number of kernel elements. The basis of Ψ has only 3 elements, which are the 3 subsets of W containing 2 points each: $M_1 = \{-1, 0\}$, $M_2 = \{-1, 1\}$, and $M_3 = \{0, 1\}$. Thus, from Theorem 10, the 3-point median of a function $f(x)$, $x \in \boldsymbol{Z}$, is equal to

$$\text{med}\left\{f(x-1), f(x), f(x+1)\right\} = \max\left\{\begin{array}{l}\min\left\{f(x-1), f(x)\right\}, \\ \min\left\{f(x-1), f(x+1)\right\}, \\ \min\left\{f(x), f(x+1)\right\}\end{array}\right\}. \quad (16)$$

Since the median operation commutes with set complementation, we can interchange min and max in (16). These max–min realizations of the median (and any other OS) provide geometrical insight for these nonlinear filters, since they involve erosions and dilations which are geometrically defined set operations.

Consider now the 1-D SP opening $\Phi(X) = X_B$ and its dual, the closing, filter $\Phi^d(X) = X^B$, where $B = \{0, 1\}$. From Theorem 11, it follows that the basis elements of Φ are the two sets $\{-1, 0\}$ and $\{0, 1\}$, and the basis elements of Φ^d are the two sets $\{0\}$ and $\{-1, 1\}$. Thus, from Theorem 9b), the 2-point opening of $f(x)$ by B can be expressed as:

$$f_B(x) = \max\left\{\min\left\{f(x-1), f(x)\right\}, \min\left\{f(x), f(x+1)\right\}\right\} \quad (17)$$

$$= \min\left\{f(x), \max\left\{f(x-1), f(x+1)\right\}\right\}. \quad (18)$$

By interchanging min and max in (17) and (18), we obtain the closing $f^B(x)$ because closing is the dual of the opening. Obviously, $f_B \leq \text{med}(f; W) \leq f^B$, as predicted by Theorem 3. Realization of the opening (closing) by the basis of its dual closing (opening) yields faster implementations of these filters, as discussed in [26] for a general window B with $|B| \geq 2$.

Example 2 (2-D Max/Median): Let $W \subseteq \boldsymbol{Z}^2$ be the 3×3 symmetric square window, and let W_1, W_2, W_3, W_4 be the 3-point subsets of W that lie on the lines passing through the center of W at slopes 0, 45, 90, 135°, respectively. Then the 2-D max/median (of window size 3) filter ψ is the pointwise maximum of the four 1-D medians by W_1, W_2, W_3, and W_4. This filter was introduced in [37] and was found to preserve edges better than the median by W. Obviously, this filter commutes with thresholding, and, hence, we can focus our analysis on the respective SP max/median filter $\Psi(X) = \cup_{i=1}^4 \Psi_i(X)$, $X \subseteq \boldsymbol{Z}^2$, where $\Psi_i(X) = \text{med}(X; W_i)$. Each filter Ψ_i has three minimal elements, i.e., the sets B_{ij}, $j = 1, 2, 3$, with $B_{ij} \subseteq W_i$ and $|B_{ij}| = 2$. Thus, ψ is the maximum of 12 2-point local minima by the sets B_{ij}, as also recently observed in [37], and the minimum of the 12 respective local maxima.

Example 3 (Linear Combination of Order-Statistics): Let $\mathfrak{F}$ be a class of real-valued sampled m-D functions closed under function translation. Let $W \subseteq \boldsymbol{Z}^m$ with $|W| = n$. Given an input function $f \in \mathfrak{F}$, the output function from the linear combination of order-statistics (LOS) filter by W is

$$[\text{LOS}(f)](x) = \sum_{k=1}^{n} a_k[\text{OS}^k(f; W)](x), \quad x \in \boldsymbol{Z}^m. \quad (19)$$

The parameters of the LOS filter are the weighting coefficients $a_k \in \boldsymbol{R}$ and the shape/size of the window W. This FP filter was introduced in [21], called "the order-statistic filter," and used for impulse noise suppression; it was also used in [22] for envelope estimation and called "order filter."

As for the morphological analysis of linear shift-in-

variant filters in Part I, we henceforth assume that 1) $a_k \geq 0\ \forall k \in \{1, 2, \cdots, n\}$, and 2) $\Sigma_{k=1}^n a_k = 1$, so that the LOS filter is increasing and TI, respectively. Then the LOS filter is also u.s.c. because it is a finite linear combination (with positive a_k's) of u.s.c. filters. If $f_{(k)} = \mathrm{OS}^k\{f(y) : y \in W\}$, $k = 1, \cdots, n$, the *kernel* of the LOS filter is the collection of all input functions f such that $[\mathrm{LOS}(f)](\mathbf{0}) \geq 0$; i.e.,

$$\mathcal{K}(\mathrm{LOS}) = \left\{ f \in \mathcal{F} : \sum_{k=1}^{n} a_k f_{(k)} \geq 0 \right\}. \tag{20}$$

From $\mathcal{K}(\mathrm{LOS})$ we can reconstruct the LOS filter exactly [26]. Moreover, we can find the set of minimal kernel functions in $\mathcal{K}(\mathrm{LOS})$, i.e., the *basis* of the LOS filter, as follows.

Theorem 12: Let $W \subseteq \mathbf{Z}^m$ with $|W| = n$, and consider $a_1, a_2, \cdots, a_n \in \mathbf{R}$ with $a_k > 0$ for each k and $\Sigma_{k=1}^n a_k = 1$. Then the basis of the LOS filter defined in (19) is equal to

$$\mathcal{B}(\mathrm{LOS}) = \left\{ g \in \mathcal{F} : \sum_{k=1}^{n} a_k g_{(k)} = 0 \right. \quad \left. \text{and} \quad g(x) = -\infty\ \forall x \notin W \right\}, \tag{21}$$

where $g_{(k)} = \mathrm{OS}^k\{g(y) : y \in W\}$, $k = 1, 2, \cdots, n$. Moreover, $\forall f \in \mathcal{F}$, $\forall x \in \mathbf{Z}^m$,

$$[\mathrm{LOS}(f)](x) = \sup_{g \in \mathcal{B}(\mathrm{LOS})} \left\{ \min_{y \in W_x} \left\{ f(y) - g(y - x) \right\} \right\}. \tag{22}$$

Proof:

a) Let $\mathcal{B}$ be the function class of (21). $\mathcal{B}$ is nonempty because $g^* \in \mathcal{B}$, where $g^*(x) = 0\ \forall x \in W$ and $g^*(x) = -\infty\ \forall x \notin W$. Assume that some $g \in \mathcal{B}$. From (20) and (21), $g \in \mathcal{K}(\mathrm{LOS})$. Let $h \in \mathcal{K}(\mathrm{LOS})$ such that $h \leq g$. Then $h(x) = g(x) = -\infty\ \forall x \notin W$, and $0 \leq [\mathrm{LOS}(h)](\mathbf{0}) \leq [\mathrm{LOS}(g)](\mathbf{0}) = 0 \Rightarrow \Sigma_{k=1}^n a_k[g_{(k)} - h_{(k)}] = 0$. Hence, $g_{(k)} = h_{(k)}\ \forall k$, because $a_k > 0$ and $g_{(k)} \geq h_{(k)}\ \forall k$, since $h \leq g$. Now, $h_{(k)} = g_{(k)}\ \forall k$ and $h \leq g \Rightarrow h(x) = g(x)\ \forall x \in W \Rightarrow h = g$. Thus, g is a minimal kernel function; i.e., $g \in \mathcal{M}$, where $\mathcal{M}$ denotes the true basis of the LOS filter. Hence, $\mathcal{B} \subseteq \mathcal{M}$.

Let now $g \in \mathcal{M}$. The g must have a minimal region of support and hence $g(x) = -\infty\ \forall x \notin W$. Further, $[\mathrm{LOS}(g)](\mathbf{0}) = p \geq 0$. If $p > 0$, the function h defined by $h(x) = -\infty\ \forall x \notin W$ and $h(x) = g(x) - p$, $x \in W$, is a kernel function (because $[\mathrm{LOS}(h)](\mathbf{0}) = 0$) with $h \leq g$ and $h \neq g$; this is a contradiction, however, because g is minimal. Hence $p = 0$ and thus $g \in \mathcal{B}$, implying that $\mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{M} \Rightarrow \mathcal{M} = \mathcal{B}$.

b) Since the LOS filter is TI increasing and u.s.c., it is equal to the supremum of FP erosions by its basis functions [26], from which (22) results. Q.E.D.

As an illustration of Theorem 12, we provide below an example where we restrict the amplitude range of all functions in $\mathcal{F}$ to be discrete, say $\mathbf{Z}$, and the coefficients a_k to be positive rational numbers. Let $W = \{-1, 0, 1\} \subseteq \mathbf{Z}$ and consider the LOS filter $[\psi(f)](x) = \Sigma_{k=1}^3 a_k[\mathrm{OS}^k(f : W)](x)$, $x \in \mathbf{Z}$, whose basis is $\mathcal{B}$. If $g \in \mathcal{B}$, let $g_1 = g(-1)$, $g_2 = g(0)$, $g_3 = g(1)$. Then, $a_1 g_{(1)} + a_2 g_{(2)} + a_3 g_{(3)} = 0$. Without loss of generality, we can assume that $g_{(1)} = 1$ [note that always $g_{(1)} \geq 0$ and $g_{(n)} \leq 0$ from (21)] because $g(x) \in \mathcal{B} \Rightarrow pg(x) \in \mathcal{B}$ for any $p \in \mathbf{N} = \mathbf{Z}_+ \cup \{0\}$. Since $a_1 + a_2 + a_3 = 1$, we must solve

$$a_2 g_{(2)} + a_3 g_{(3)} = a_2 + a_3 - 1 \tag{23}$$

subject to

$$1 \geq g_{(2)} \geq 1 - \frac{1}{a_2 + a_3} \geq g_{(3)}, \tag{24}$$

where $g_{(2)}$ and $g_{(3)}$ are the integer unknowns. To each solution $(g_{(1)}, g_{(2)}, g_{(3)})$ there corresponds a multitude of 3-tuples (g_1, g_2, g_3), whose number is equal to the number of distinct permutations of the 3-tuple $(g_{(1)}, g_{(2)}, g_{(3)})$. The solutions of (23) can be obtained from a search of the finite region of $\mathbf{Z}^2$ delineated by the constraints (24). For example, let $a_2 = 2/4$ and $a_1 = a_3 = 1/4$. Then, the solutions are

$(g_{(1)}, g_{(2)}, g_{(3)})$	$(g_1 = g(-1), g_2 = g(0), g_3 = g(1))$
$(1, 0, -1)$	$(1, 0, -1), (0, 1, -1), (-1, 0, 1), (1, -1, 0), (-1, 1, 0), (0, -1, 1)$
$(1, 1, -3)$	$(1, 1, -3), (-3, 1, 1), (1, -3, 1)$

Let $\mathcal{L}$ be the set of the 9 basis functions defined by (g_1, g_2, g_3) in the above table. Then $\mathcal{B}(\psi) = \{pg(x) : p \in \mathbf{N}, g(x) \in \mathcal{L}\}$. Thus,

$$\left[\sum_{k=1}^{3} a_k \mathrm{OS}^k(f; W) \right](x) = \sup_{p \in \mathbf{N}} \left\{ \max_{g \in \mathcal{L}} \left[\min_{y \in W_x} \left\{ f(y) - pg(y - x) \right\} \right] \right\}.$$

Finally, it is straightforward to extend the above procedure to LOS filters with $n > 3$.

VI. Relations Between Morphological and Stack Filters

Before we discuss the relations between morphological and stack filters, a few definitions are needed from the theory of *Boolean functions* [30], [31]. Any Boolean expression of n variables $x_1, x_2, \cdots, x_n \in \{0, 1\}$ can be written as Boolean sum-of-product (SOP) terms or as Boolean product-of-sum (POS) terms. Each product or sum term may contain each *literal* (a variable x_i or its complement x_i') at most once and/or the Boolean constants 0 or 1. To each Boolean expression there corresponds a unique Boolean function $\beta(\mathbf{x}) \in \{0, 1\}$, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. A Boolean function is usually described through a *truth table*. Two Boolean expressions

are called *equivalent* if they correspond to the same Boolean function. A Boolean function γ is said to *imply* β iff $\beta(\boldsymbol{x}) = 1$ for each $\boldsymbol{x}$ such that $\gamma(\boldsymbol{x}) = 1$. A *prime implicant* π of β is a product term which implies β, such that deletion of any literal from π results in a new product which does not imply β. A *prime implicate* of β is a sum term σ implied by β, such that deletion of any literal from σ results in a new sum term which is not implied by β. Any *minimal* SOP (respectively, POS) expression for β is a sum (respectively, product) of prime implicants (respectively, prime implicates) such that removal of any of them makes the remaining expression no longer equivalent to β, and the expression contains the minimum number of literals and product (respectively, sum) terms. This minimal expression is not necessarily unique. A function $\beta(\boldsymbol{x})$ is called *positive* if it can be represented by an SOP or POS expression in which no variable appears in uncomplemented form. Each positive function has a unique minimal SOP expression that is positive and is the sum of all its prime implicants; it also has a unique minimal POS expression that is positive and is the product of all its prime implicates.

A. Stack Filters

Wendt *et al.* [27], [28] defined the stack filters as follows. Consider an input function $f(z)$, $z \in \boldsymbol{Z}^m$, with a finite number M of amplitude levels. Threshold f at all amplitudes $t \in \{0, 1, \cdots, M-1\}$, obtain its cross sections $X_t(f)$, and consider their respective characteristic functions $\chi_{X_t(f)}(z)$, or simply, $\chi_t(z)$. Filter all binary signals χ_t by a TI, discrete, *increasing*, *binary* filter Φ_b. (In [27] and [28], the increasing property of Φ_b is called "stacking property.") Then $t_1 \le t_2 \Rightarrow \chi_{t_1} \ge \chi_{t_2} \Rightarrow \Phi_b(\chi_{t_1}) \ge \Phi_b(\chi_{t_2})$. Assume that at each $z \in \boldsymbol{Z}^m$, the value of the output $[\Phi_b(k)](z)$ is determined only from the values of the input signal $k(z)$ inside a fixed finite window W of n points shifted at location z. Since Φ_b is a TI and binary filter, its defining rule can be represented by the *truth table* of a Boolean function $\beta(x_1, x_2, \cdots, x_n)$, where the variables x_i represent the n values $k(y)$, $y \in W_z = \{z + w : w \in W\}$, of the input signal $k(z)$ inside W_z. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ represent two vectors in $\{0, 1\}^n$, and define a partial ordering relation $\preceq$ in $\{0, 1\}^n$ through the rule $\boldsymbol{x} \preceq \boldsymbol{y} \Leftrightarrow x_i \le y_i \ \forall i$. The fact that Φ_b is an increasing filter is equivalent to the condition $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow \beta(\boldsymbol{x}) \le \beta(\boldsymbol{y}) \ \forall \boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n$, which in turn is equivalent to β being a positive Boolean function [29]. Finally, the multilevel function

$$[\mathrm{ST}_\beta(f)](z) = \sum_{t=0}^{M-1} [\Phi_b(\chi_{X_t(f)})](z), \quad z \in \boldsymbol{Z}^m, \tag{25}$$

is viewed as an output function for each input $f(z)$, and thus, the *stack filter* ST_β is defined. By varying β, or equivalently Φ_b, a different stack filter is obtained.

As an aside, for input functions f whose amplitude range is a continuum, say $\boldsymbol{R}$, we can extend the definition (25) as

$$[\mathrm{ST}_\beta(f)](z) = \int_{\boldsymbol{R}} [\Phi_b(\chi_{X_t(f)})](z)\, dt. \tag{26}$$

B. Morphological Analysis of Stack Filters

Any stack filter ST_β, by its construction, is a TI discrete FSP filter commuting with thresholding; hence it is increasing and u.s.c. too (due to Theorem 3 of Part I). To each binary filter Φ_b, we can uniquely associate an SP filter Φ, such that whenever Φ operates on a set $A \subseteq \boldsymbol{Z}^m$, Φ_b operates on the characteristic function χ_A of A, and vice versa. Φ is TI and increasing iff Φ_b is TI and increasing; further, Φ is u.s.c. due to the *finite* window W. Then Φ is the respective SP filter of the FSP filter ST_β, and from our discussion in Section II-D of Part I,

$$[\mathrm{ST}_\beta(f)](z) = \sup\left\{t \in \boldsymbol{V} : z \in \Phi[X_t(f)]\right\}, \tag{27}$$

where $\boldsymbol{V}$ can be either $\boldsymbol{R}$ or $\boldsymbol{Z}$ and $X_t[\mathrm{ST}_\beta(f)] = \Phi[X_t(f)] \ \forall t \in \boldsymbol{V}$. Thus, the *sum*-definition of ST_β in (25) is only a special case of its *sup*-definition in (27). For a finite number of amplitude levels t, the supremum becomes a maximum; in [28] and [38], the usefulness of this *max*-definition of stack filters is recognized for fast VLSI implementations.

For the cases when $\beta(x_1, \cdots, x_n)$ is a *threshold function* [30], Wendt *et al.* [28] provided a functional definition for ST_β as a generalization of OS filters, in which multiple repetitions of the same element are allowed. In [27], [28], all the stack filters corresponding to the 20 positive Boolean functions of $n = 3$ variables were examined. To obtain a functional definition of ST_β from a threshold function would be inefficient for large n because of the large number of repetitions of the same elements and because deriving the threshold function form a large truth table is not a simple task. In addition, for $n > 3$, not all positive Boolean functions are threshold functions. For example, in [28] it was observed that, of the 7581 positive functions with $n \le 5$, only 3287 are threshold functions; hence, it was conjectured [28] that there are many stack filters whose outputs cannot be expressed as simple functions of the input samples. However, by using morphological concepts, we provide below a general algorithm that obtains from the SOP or POS positive Boolean expressions of β the functional definition of *any* stack filter ST_β in terms of max–min operations.

A) Let $W \subseteq \boldsymbol{Z}^m$ be the window of n points associated with $\beta(x_1, x_2, \cdots, x_n)$ and let $I(\cdot)$ be an index function that assigns to each $w \in W$ a unique integer $I(w)$ in $\{1, 2, \cdots, n\}$. For example, let $m = 1$, $n = 3$, $W = \{-1, 0, 1\}$, and $I(w) = w + 2$, $w \in W$.

B) Obtain the minimal SOP and POS expressions for β; e.g.,

$$\beta(x_1, x_2, x_3) = x_1x_2 + x_2x_3 = x_2(x_1 + x_3). \tag{28}$$

C) Obtain the respective SP filter Φ operating on an input set S by replacing: the Boolean sum/product (logical OR/AND) with union/intersection, respectively; and each variable x_i in β by a translation S_{-w} of S, where $w \in W$

and $I(w) = i$. For example,

$$\Phi(S) = (S \cap S_1) \cup (S \cap S_{-1}) = S \cap (S_1 \cup S_{-1}). \tag{29}$$

D) Obtain the respective stack filter ST_β from Φ by replacing: the set S in C) with the cross sections $X_t(f)$ of an input function $f(z)$; the finite union/intersection of cross sections with pointwise max/min, respectively, of functions; and cross-section translations $[X_t(f)]_{-w}$ with function shifts $f(z + w)$. For the example of (29), $\mathrm{ST}_\beta(f)$ assumes a max–min and a min–max functional definition identical to the realizations of the opening f_B, $B = \{0, 1\}$, in (17) and (18). Moreover, with the hindsight obtained, we can combine steps C) and D) in a single step as follows.

C*) Obtain the functional definition of ST_β from β by replacing: each Boolean sum/product with a max/min, respectively; and each variable x_i by a shifted version $f(z + w)$ of the input function $f(z)$, where $w \in W$ and $I(w) = i$.

Concluding, each stack filter can be expressed as a finite max–min or min–max operation. Conversely, any finite max–min or min–max operation is TI and commutes with thresholding [23]; hence it corresponds to a TI increasing u.s.c. SP filter, or equivalently to a positive Boolean function, which in turn defines a stack filter. Therefore, the stack filters are the *class of all discrete FSP filters that are TI, commute with thresholding, and can be expressed as a finite maximum of local minima or as a minimum of local maxima.* Thus, discrete FSP erosions, dilations, closings, openings, open–closings, medians, and OS filters, as well as any finite cascade or parallel (using pointwise min–max) combination of these filters, are all special cases of stack filters. However, the general FP erosion, dilation, opening, or closing of a function f by another nonbinary structuring function g (defined in Part I), as well as any finite cascade or parallel combination of these FP filters do *not* commute with thresholding and, hence, they are *not* stack filters. Moreover, these latter morphological filters include the stack filters as a special case because they become stack filters whenever all the structuring functions involved in their definition become binary. Finally, the original definition of stack filters in (25) via Boolean functions allows only for discrete filters, whereas our definition in (27) via TI increasing u.s.c. SP filters allows for both discrete and analog filters.

C. Minimal Elements of Stack Filters

Theorem 9 applies to any stack filter ST_β and its respective SP filter Φ. That is, ST_β can be represented as a supremum of FSP erosions by all the basis sets of Φ and also as an infimum of FSP dilations by all the basis sets of its dual SP filter Φ^d. Next we will establish some connections between these ideas and the representation of the positive Boolean function β by a minimal SOP and POS expression.

In $\{0, 1\}^n$, a vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is called a *minimal true vector* of β iff $\beta(\boldsymbol{x}) = 1$ and $\boldsymbol{x}$ is not preceded (with respect to the vector ordering $\preceq$) by any other vector $\boldsymbol{v}$ with $\beta(\boldsymbol{v}) = 1$. A vector $\boldsymbol{y} \in \{0, 1\}^n$ is called a *maximal false vector* of β iff $\beta(\boldsymbol{y}) = 0$ and $\boldsymbol{y}$ is not followed (with respect to $\preceq$) by any other vector $\boldsymbol{u}$ with $\beta(\boldsymbol{u}) = 0$. The Boolean function $\beta^d(\boldsymbol{x}) = \beta'(\boldsymbol{x}')$, $\boldsymbol{x}' = (x_1', x_2', \cdots, x_n')$, is called the *dual* function of β. Since β is positive, β^d is positive too. Hence, to β^d there corresponds a unique TI increasing u.s.c. SP filter which is the dual Φ^d of Φ. In addition, we have the following.

Theorem 13: Let ST_β be an m-D stack filter, whose defining rule is associated with a fixed window W of n points. Let Φ be its respective SP filter, whose dual SP filter is Φ^d. Consider the positive Boolean function $\beta(x_1, x_2, \cdots, x_n)$ corresponding to Φ and its dual function β^d corresponding to Φ^d. Let $I: W \to \{1, 2, \cdots, n\}$ be a one-to-one index function. Then the following is true.

a) The stack filter ST_β can be expressed as a finite pointwise maximum of moving local minima and also as a finite minimum of moving local maxima.

b) The stack filter ST_{β^d} defined by β^d can be obtained from ST_β by interchanging max with min.

c) To each minimal true vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ of β, there corresponds a unique minimal kernel element $G = \{w \in W : a_{I(w)} = 1\}$ of Φ, and vice versa. To each SP erosion by G, there corresponds a unique prime implicant of β.

d) To each minimal true vector $\boldsymbol{b} = (b_1, \cdots, b_n)$ of β^d, and, equivalently, to each maximal false vector of β, there corresponds a unique minimal kernel element $H = \{w \in W : b_{I(w)} = 1\}$ of Φ^d, and vice versa. To each SP dilation by H, there corresponds a unique prime implicant of β^d, and, equivalently, a unique prime implicate of β.

Proof:

a) It was shown in Section VI-B.

b) The function β^d is obtained from β by interchanging Boolean sums with products and 1 with 0 (De Morgan's laws). The filter ST_{β^d} is obtained from β^d by replacing Boolean sum/product with max/min. Hence, ST_{β^d} can be obtained from ST_β by interchanging max with min.

c) Φ has at least one basis set because Φ is TI increasing and u.s.c. [26]. The subset G of W belongs to the kernel of Φ, because $\beta(\boldsymbol{a}) = 1 \Rightarrow \boldsymbol{0} \in \Phi(G_x)$ for some $x \in \boldsymbol{Z}^m$, and since Φ is TI we can assume without loss of generality that $x = \boldsymbol{0}$. Since $I(\cdot)$ is one-to-one, the correspondence $\boldsymbol{a} \leftrightarrow G$ is one-to-one. Hence, G is minimal with respect to set inclusion since $\boldsymbol{a}$ is minimal with respect to vector ordering $\preceq$. From the vector $\boldsymbol{a}$ we can obtain the product $p = x_{i_1} x_{i_2} \cdots x_{i_k}$ where $i_r \in \{1, \cdots, n\}$ for $r = 1, \cdots, k \le n$, and p contains the variable x_j iff $a_j = 1$. The correspondence $a \leftrightarrow p$ is one-to-one because p contains no complemented variables. Since $\boldsymbol{a}$ is a minimal true vector of β, p is a prime implicant of β. The positive Boolean function β can be expressed as the sum of all its prime implicants, which are all positive. Equivalently, the TI increasing u.s.c. SP filter Φ can be ex-

pressed as the union of SP erosions by all its basis sets [26]. Thus, all the correspondences $p \leftrightarrow \boldsymbol{a} \leftrightarrow G \leftrightarrow \Psi(X) = X \ominus G^s = \bigcap_{w \in G} X_{-w}$ are one-to-one. Hence, the prime implicant p corresponds to the SP erosion by G by replacing each x_j in p with X_{-w}, where $I(w) = j$, and the Boolean product with $\cap$.

d) From the proof of part c), the correspondences between the minimal true vectors $\boldsymbol{b}$ of the positive Boolean function β^d, the basis sets of Φ^d (which exist for the same reasons as for Φ), and the prime implicants of β^d are all one-to-one. The vector $\boldsymbol{b}$ is the dual of a maximal false vector of β because $\beta^d(\boldsymbol{b}) = 1 \Leftrightarrow \beta(\boldsymbol{b}') = 0$. Equivalently, each prime implicant of β^d is the dual of a prime implicate of β, and the minimal SOP expression for β^d is the dual of the minimal POS expression for β. Thus, the subset H of W corresponds to the prime implicate $s = x_{i_1} + \cdots + x_{i_k}$ of β, where $i_r \in \{1, \cdots, n\}$ for $r = 1, \cdots, k \le n$, and s contains the variable x_j iff $b_j = 1$. Hence, s yields an SP dilation filter $\Psi(X) = X \oplus H^s = \bigcup_{w \in H} X_{-w}$ by replacing each x_j in s with X_{-w}, where $I(w) = j$, and the Boolean sum with $\bigcup_{w \in H}$. Q.E.D.

The following example clarifies Theorem 13 and its proof.

Example 4: Consider the 1-D FSP opening $\phi(f) = f_A$ where $A = \{-1, 0, 1\}$. Thus, $f_A(x) = \max\{g(x-1), g(x), g(x+1)\}$, where $g(x) = \min\{f(x-1), f(x), f(x+1)\}$, $x \in \boldsymbol{Z}$. Its respective SP filter is $\Phi(X) = X_A$, $X \subseteq \boldsymbol{Z}$, and its dual SP filter is the closing $\Phi^d(X) = X^A$. From Theorem 9a), $\forall X \subseteq \boldsymbol{Z}$,

$$X_A = \bigcup_{G \in \mathcal{B}(\Phi)} X \ominus G^s = \bigcap_{H \in \mathcal{B}(\Phi^d)} X \oplus H^s, \quad (30)$$

where the basis of Φ and Φ^d can be found from Theorem 11. That is, $\mathcal{B}(\Phi) = \{\{-2, -1, 0\}, \{-1, 0, 1\}, \{0, 1, 2\}\}$, and $\mathcal{B}(\Phi^d) = \{\{0\}, \{-2, 1\}, \{-1, 1\}, \{-1, 2\}\}$. From Theorem 9b) we obtain a functional definition for ϕ as [set $f_i = f(x - 3 + i)$, $i = 1, 2, 3, 4, 5$]

$$f_A(x) = \max\{\min(f_1, f_2, f_3), \min(f_2, f_3, f_4), \min(f_3, f_4, f_5)\} \quad (31)$$

$$= \min\{f_3, \max(f_1, f_4), \max(f_2, f_4), \max(f_2, f_5)\}. \quad (32)$$

Since ϕ is a stack filter, we can also obtain the above max–min and min–max definitions for ϕ from its respective Boolean function. That is, let β and its dual β^d be the positive Boolean functions corresponding to the increasing SP filters Φ and Φ^d, respectively. The window associated with ϕ, Φ, Φ^d is $W = A \oplus A^s = \{-2, -1, 0, 1, 2\}$. Thus, β and β^d will be functions of 5 variables x_1, x_2, x_3, x_4, x_5, where the index function is $I(w) = w + 3$, $w \in W$. Next we summarize how to obtain β and β^d from the basis of Φ and Φ^d, by using Theorem 13c) and d).

Basis Sets of Φ	Basis Sets of Φ^d
$\{-2, -1, 0\}, \{-1, 0, 1\}, \{0, 1, 2\}$	$\{0\}, \{-2, 1\}, \{-1, 1\}, \{-1, 2\}$
min. true vectors of β	min. true vectors of β^d
$(1, 1, 1, 0, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$	$(0, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 1)$
prime implicants of β	prime implicants of β^d
$x_1x_2x_3, x_2x_3x_4, x_3x_4x_5$	$x_3, x_1x_4, x_2x_4, x_2x_5$
prime implicates of β^d	prime implicates of β
$x_1 + x_2 + x_3, x_2 + x_3 + x_4, x_3 + x_4 + x_5$	$x_3, x_1 + x_4, x_2 + x_4, x_2 + x_5$

Each of the positive Boolean functions β and β^d can be expressed now in minimal SOP (POS) form as the sum (product) of its prime implicants (implicates). For example,

$$\beta(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 \quad (33)$$

$$= x_3(x_1 + x_4)(x_2 + x_4)(x_2 + x_5). \quad (34)$$

By replacing Boolean sum with max and product with min, we obtain from (33) the max–min definition of the FSP opening in (31), and from (34) its min–max definition in (32).

Note that if we exchange in all of the discussion of Example 4 the roles of Φ and Φ^d or, equivalently, the roles of β and β^d, we obtain the max–min and min–max definitions of the dual stack filter of ϕ, i.e., the closing $f \to f^A$, by replacing the roles of max and min in (31) and (32).

The 3-point opening f_A and closing f^A are comparable to the 5-point median med $(f; W)$, as Section IV explains. From Theorem 10, the SP median by W has 10 basis sets, i.e., all the 3-point subsets of W, and is identical to its dual filter. Hence, its respective positive Boolean function has a unique minimal expression as the sum of the 10 prime implicants $x_ix_jx_k$, where $i, j, k \in \{1, 2, 3, 4, 5\}$ and $i \ne j \ne k$.

Concluding, the two approaches of representing stack filters either by their basis (minimal kernel elements) or by their Boolean functions in minimal SOP or POS forms have many analogies, which are summarized by Theorem 13. In addition, the following comparisons are evident. 1) For small windows, e.g., of 3 points, both approaches are relatively simple to apply. 2) For large windows, e.g., with number of points ≥ 5, to find minimal SOP or POS expressions for Boolean functions is a computationally complex task and has become the objective of extensive research in the field of switching theory. By contrast, for stack filters (of any window size) that are parallel or cascade combinations of median, OS filters, erosions, dilations, openings, or closings, we can find their basis immediately from Theorems 10 and 11 and hence their max–min definition from Theorem 9. In addition, from the basis we obtain the Boolean functions in minimal SOP or POS expressions as a byproduct. 3) For large windows and arbitrary stack filters whose definition does not allow

easily to find their basis, the Boolean function seems at first more helpful since there are standard algorithms [30], [31] to find its minimal SOP or POS forms. In these cases, the Boolean function approach can also be seen as an algorithm (together with Theorem 13) to find the basis.

VII. Conclusions

We have extended the theory of median, OS, and stack filters by introducing the use of mathematical morphology to analyze them and to relate them with morphological filters. OS filters are both function- and set-processing (FSP). Using morphological concepts, we showed that OS filters commute with thresholding. This fact allows us to analyze OS filters by focusing only on OS of sets. OS of sets are easier to analyze and implement because their definition involves *counting* of points, in contrast to the *sorting* of numbers required for OS of functions.

We have shown that each kth OS filter by a window W of n points is equal to the maximum of the local minima (erosions) by all k-point subsets of W, and to the minimum of the local maxima (dilations) by all $(n - k + 1)$-point subsets of W; this representation is given by a closed formula which does not involve sorting. We proved this max–min representation of OS filters by using first a combinatorial proof and second their kernel and basis (minimal kernel elements) representation. The *minimal kernel elements* of the kth OS filter by W are all the k-point subsets of W. The minimal elements approach allows us to unify OS filters together with a large class of linear and nonlinear filters which can be expressed as a supremum or union of erosions.

We have found that medians, openings, and closings are closely related. If W is a 1-D convex symmetric $(2n + 1)$-point window and B is a 1-D convex $(n + 1)$-point set, the median (and its iterations) of any signal by W is bounded below by the opening and above by the closing by B. If a signal is a root of both the opening and closing by B, then it is a root of the median by W. (The converse is true only for signals of finite extent.) Hence, median roots with respect to W can be obtained by finding signals that are roots of both the opening and the closing by B. For example, the open–closing or clos–opening by B provide roots of the median by W in a single pass. Moreover, the median root obtained through iterating the median filter by W on a finite extent signal is bounded from below and above, respectively, by the open–closing and clos–opening by B; it was also experimentally observed that this median root lies close to the open–closing and clos–opening. These results combined with the fact that open–closing requires comparable or less computational complexity than a single median, (and, hence, much less complexity than iterating the median,) makes the open–closing (or clos–opening) more appealing. For suppressing impulse noise in signals, the open–closing behaves very similarly to the median; in addition, it can discriminate between positive and negative noise spikes, whereas the median cannot. Some similar results were obtained for 2-D filtering.

Finally, we have shown that the stack filters, whose original definition [27], [28] was based on positive Boolean functions, are actually the class of all finite maxima of local minima and minima of local maxima filters. As such, they contain all median and OS filters, and *only* those FSP morphological filters that commute with thresholding. Stack filters can be expressed as minimal forms of max–min operations based either on irreducible forms of their Boolean functions or on their minimal kernel elements. We have established the theoretical equivalence of both of these approaches and provided a systematic algorithm to find the max–min expression of any stack filter from its Boolean function.

Thus, FSP erosion and dilation are the prototypes for representing any median, OS, or stack filter. Since all these filters commute with thresholding and set erosion/dilation can be implemented using an intersection/union of shifted versions of the input set, the erosion/dilation (min/max) representation of OS and stack filters suggests simple methods for their *parallel* implementation.

In short, mathematical morphology combined with the minimal elements representation provides a self-contained mathematical framework that, based on simple concepts, facilitates the theoretical analysis of all the above nonlinear filters, establishes their interrelationships, suggests methods for their implementation, and further relates them to a large class of nonlinear filters, linear filters, and algorithms for shape analysis [26], [39].

Acknowledgment

We wish to thank an anonymous reviewer for many suggestions that helped us to improve the contents of this paper, and another reviewer whose helpful comments motivated the inclusion of the section on stack filters.

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Petros Maragos (S'80–M'85), for a photograph and biography, see this issue, p. 1169.

Ronald W. Schafer (S'62–M'67–SM'74–F'77), for a photograph and biography, see this issue, p. 1169.

Corrections to "Morphological Filters—Parts I and II"

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In the above papers,[1] the following should be corrected.

PART I

p. 1156: In Table I, "min $f(x)$" and "pin $f(x)$" should read "*minf* (x)" and "*pinf* (x)," respectively, which are abbreviations of minus and plus infinity.

Equation (11) should read

$$[\phi(f)](x) = \sup\{t \in V : x \in \Phi[X_t(f)]\}. \tag{I.11}$$

PART II

p. 1172: For (5) it is assumed that $f(z) \geq 0 \ \forall z$, and hence (5) should read

$$[\mathrm{OS}^k(f; W)](z) = \int_0^\infty \chi_{\mathrm{OS}^k[X_t(f); W]}(z)\, dt. \tag{II.5}$$

In the proof (13th line) of Theorem 1, replace "$T_m \neq T_1$" with "T_m, $m \neq 1$."

p. 1175: Line 16 (left column) should read

$$X_B = \cdots 00000000 \cdots$$

p. 1176: Equation (14) should read

$$z \in X_H \Rightarrow |H_z \cap X| \geq 4. \tag{II.14}$$

In the proof (1st line) of Theorem 8, replace "$x \in \mathrm{med}(X; H)$" with "$z \in \mathrm{med}(X; H)$."

p. 1177: In line 24 (left column), replace "$\Psi^d(X) = [(X^c)]^c \cdots$" with "$\Psi^d(X) = [\Psi(X^c)]^c \cdots$."

p. 1178: In Theorem 11(b), replace "$\cdots 0 \in \{a, b\}^A$" with "$\cdots 0 \in \{a, b\}^A$ and $a < 0 < b$."

p. 1179: In line 5 (right column), "Without loss . . . assume" should be replaced by "For simplicity, assume first." Further, starting from "Let $\mathcal{L}$ . . . ," lines 25–29 (right column) should be replaced by the following text:

When $g_{(1)} = p = 0, 1, 2, \cdots$, (23) and (24) generalize to

$$a_2 g_{(2)} + a_3 g_{(3)} = -pa_1 \tag{23a}$$

$$\text{s.t.} \quad p = g_{(1)} \geq g_{(2)} \geq \frac{-pa_1}{a_2 + a_3} \geq g_{(3)}. \tag{23b}$$

For each p, let $\mathcal{L}_p$ be the finite set of basis functions defined by all tuples (g_1, g_2, g_3) corresponding to the solutions $(g_{(1)}, g_{(2)}, g_{(3)})$ of (23a) with the constraint (23b). Then the LOS filter ψ has a basis $\mathcal{B}(\psi) = \cup_{p \in N} \mathcal{L}_p$ and admits a representation as in (22).

p. 1180: For (26) it is assumed that $f(z) \geq 0 \ \forall z$, and hence (26) should read

$$[ST_\beta(f)](z) = \int_0^\infty [\Phi_b(\chi_{X_t(f)})](z)\, dt. \tag{II.26}$$

In line 30 (right column) replace "form" with "from."

p. 1182: In the table at the top (right column), the minimal true vectors of β^d are four; the missing vector is (0, 1, 0, 1, 0).

Manuscript received August 20, 1988.

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IEEE Log Number 8826093.

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