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EVOLUTION EQUATIONS FOR CONTINUOUS-SCALE MORPHOLOGY

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ABSTRACT

Multiscale signal analysis has recently emerged as a useful framework for many computer vision and signal processing tasks. Morphological filters can be used to develop nonlinear multiscale operators which have certain advantages over linear multiscale approaches in that they preserve important signal features such as edges. In this paper we discuss several nonlinear partial differential equations that model the scale evolution associated with continuous-space multiscale morphological erosions, dilations, openings, and closings. These systems relate the infinitesimal evolution of the multiscale signal ensemble in scale space to a nonlinear operator acting on the space of signals. The type of this nonlinear operator is determined by the shape and dimensionality of the structuring element used by the morphological operators, generally taking the form of nonlinear algebraic functions of certain differential operators.

1. INTRODUCTION

Both in computer vision and video data compression, important problems such as feature and/or motion detection and frequency multi-band analysis have recently been addressed using *multiscale image analysis*. In most of the work in this area the multiscale versions of an image have been obtained by acting on the image with a *linear* smoothing filter whose impulse response is a Gaussian $G_s(x) = (1/\sqrt{4\pi s})\exp[-x^2/(4s)]$ with variance proportional to scale s. For computer vision tasks, Witkin [11] proposed a continuous (in scale s and signal argument x) multiscale signal ensemble $\gamma(x,s) = f(x) * G_s(x)$, where an original signal f is convolved with a multiscale Gaussian. It is well known, e.g. see [2], that γ can be generated from the diffusion equation $\partial\gamma/\partial s = \partial^2\gamma/\partial x^2$, starting from the initial condition $\gamma(x,0) = f(x)$. This partial differential equation (PDE) represents a continuous dynamical system that generates this multiscale evolution of f. In [7, 3] there are nonlinear refinements of this idea. Despite the mathematical tractability of the linear multiscale approaches, there is a variety of *nonlinear* smoothing filters, including the morphological openings and closings [6, 8, 9, 4] and the anisotropic and nonlinear diffusion schemes [7, 3], that can provide a multiscale image ensemble and have the advantage over the linear Gaussian smoothers that they do not blur or shift image edges. This attractive property of morphological smoothers is illustrated in Figure 1.

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In this paper we study multiscale morphological filters. Multiscale openings and closings of binary images were first developed by Matheron [6] in his theory of size distributions. They have been used extensively in image analysis and applications of mathematical morphology to biology and petrography [8], for multiscale shape description and representation via skeleton transforms [4], and for signal smoothing/reconstruction in multiresolution morphology [1]. The use of morphological filters for multiscale signal analysis is not limited to operations of a smoothing type. For instance, in fractal image analysis, morphological erosion and dilation filters can provide multiscale scale distributions of the shrink-expand type from which the fractal dimension can be computed [5].

Motivated by the wide applicability of multiscale morphological smoothing, as well as by the potential applications of continuous dynamical systems to analog VLSI and neural networks, in this paper we develop nonlinear PDEs that model multiscale morphological filters of the shrink-expand type (erosions/dilations) and of the smoothing type (openings/closings) as dynamical systems. Since the basic ingredients of multiscale morphology are multiscale erosions and dilations, the biggest part of our analysis focuses on deriving the nonlinear PDEs modeling the scale evolution of a variety of multiscale erosions and dilations. These PDEs are nonlinear algebraic functions of first-order differential operators, and their form varies according to the shape and dimensionality of the structuring element. Overall, our work can be viewed as describing a nonlinear scale space based on min-max operators rather than being an extension of Gaussian convolutions.

2. DILATIONS AND EROSIONS

Henceforth, let $f : \mathbf{R}^\nu \to \mathbf{R}$ be a continuous function representing some ν–D signal and let the continuous *'structuring function'* $g : B \to \mathbf{R}$ represent some structuring element with a compact support $B \subseteq \mathbf{R}^\nu$. Let also $f \oplus g$ and $f \ominus g$ denote the morphological dilation and erosion of f by g. We define the *multiscale* dilations and erosions of f by g at scale $s \geq 0$ as the functions

$$\alpha(x,s) \triangleq f \oplus g_s(x) = \sup\{f(x-v) + sg(v/s) : v \in sB\}$$

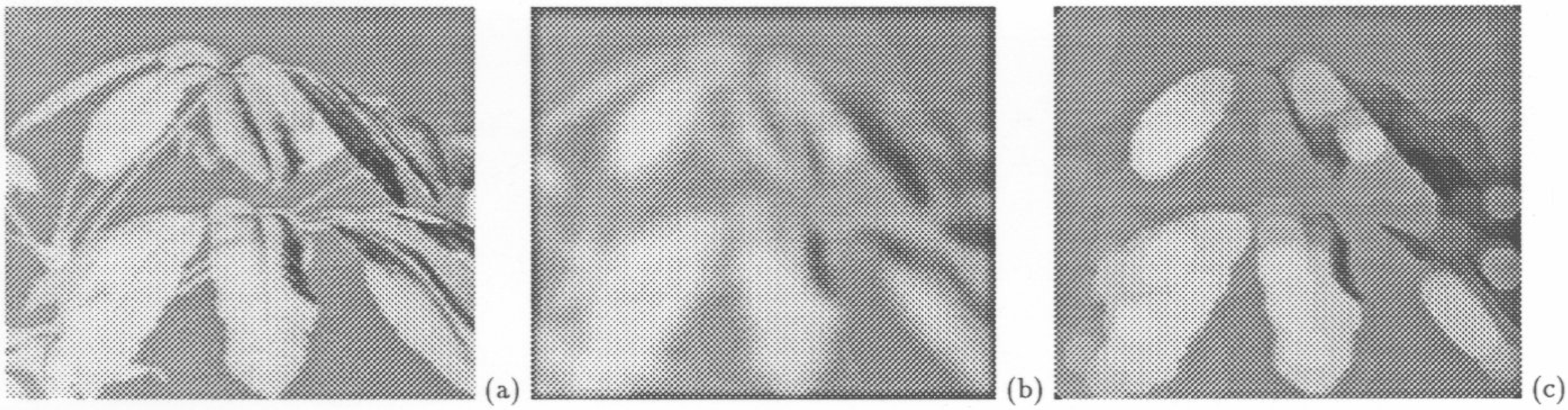

Figure 1: (a) Original image. Smoothing (b) via Gaussian convolution and (c) via opening by an octagon at same scales.

$$\beta(x,s) \quad \triangleq \quad f \ominus g_s(x) = \inf\{f(x+v) - sg(v/s) : v \in sB\}$$

where $s \geq 0$ is the continuous scale parameter and $g_s : sB \to \mathbb{R}$ is a scaled version of g defined as $g_s(x) = sg(x/s)$ for $s > 0$ with support $sB = \{sb : b \in B\}$. (If $s = 0$, then $sB = \{0\}$ and $g_0(0) = 0$.) The function g_s has the same shape as g but both its domain and range are scaled by a factor s.

If $g(x) = 0$ for all $x \in B$, then the dilation and erosion by g_s become a moving sup and inf of the input signal inside the moving window set sB. We refer to these operations as *multiscale dilations/erosions by a structuring set* (B), because the zero-valued g carries the same information with its support set B. We dedicate a significant part of this paper to the analysis of such morphological operators that use structuring sets, because they are simpler to analyze and implement than their counterparts that use structuring functions and have found more applications.

Our primary goal is to attempt to make sense out of the following evolution equation

$$\frac{\partial \alpha}{\partial s}(x,s) \triangleq \lim_{r \downarrow 0} \frac{\alpha(x,s+r) - \alpha(x,s)}{r}$$

for α and to interpret its solution in morphological terms. Toward this goal, we henceforth constrain g to be *non-negative* and $\cap$*-convex*; hence, B is convex too. Convexity of B endows the set family $\{sB : s \geq 0\}$ with a semigroup structure [6, 9]; i.e., $sB \oplus rB = (s+r)B$ and similarly for g_s. Then the serial composition laws of dilation and erosion cause the multiscale dilation and erosion operators $\mathcal{D}_s(f) = f \oplus g_s$ and $\mathcal{E}_s(f) = f \ominus g_s$ to inherit this *semigroup* structure:

$$\mathcal{D}_s(\mathcal{D}_r(f)) = \mathcal{D}_{s+r}(f) \quad ; \quad \mathcal{E}_s(\mathcal{E}_r(f)) = \mathcal{E}_{s+r}(f)$$

It follows from this semigroup structure that

$$\frac{\partial \alpha}{\partial s} = \lim_{r \downarrow 0} \frac{\sup_{v \in rB}\{\alpha(x-v,s) + rg(v/r)\} - \alpha(x,s)}{r}$$

and similarly for β using erosion.

2.1. Dilations/Erosions by 1–D Sets

Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, let $B = [-1,1]$, assume a constant $g : B \to \{0\}$, and let $\alpha(x,s) = f \oplus sB$ and $\beta(x,s) = f \ominus sB$. Then the following result yields PDEs for α and β. Although we state these results in *local* form, the issues relating to existence of solutions can be thought of as having been resolved since α and β are well defined for all (x,s).

THEOREM 1 . *If the partial derivatives* $\partial a/\partial x$ *and* $\partial \beta/\partial x$ *exist at some point* x *and scale* s*, then*

$$\frac{\partial \alpha}{\partial s}(x,s) = \left|\frac{\partial \alpha}{\partial x}(x,s)\right| \quad ; \quad \frac{\partial \beta}{\partial s}(x,s) = -\left|\frac{\partial \beta}{\partial x}(x,s)\right|$$

Proof: Note that $\alpha(x+v,s) - \alpha(x,s) = \alpha_x v + |v| o(v)$ where $\alpha_x = \partial\alpha/\partial x$ and $o(v) \to 0$ as $v \to 0$. Thus, by ignoring the term with $o(v)$ in the limit $r \downarrow 0$, it follows that

$$\frac{\partial \alpha}{\partial s} = \lim_{r \downarrow 0} \frac{\sup\{\alpha_x v : |v| \leq r\}}{r} = \lim_{r \downarrow 0} \frac{|\alpha_x| r}{r} = \left|\frac{\partial \alpha}{\partial x}\right|$$

Similarly for β by replacing sup with inf. □

Thus, assuming that the partial derivatives $\partial\alpha/\partial x$ and $\partial\beta/\partial x$ are continuous, these two nonlinear PDEs can generate the 1–D multiscale dilations and erosions starting from the initial conditions $\alpha(x,0) = \beta(x,0) = f(x)$. However, even if f is differentiable, as the scale s increases the multiscale erosions/dilations can create discontinuities in their derivatives $\partial/\partial x$; then these derivatives and the generator PDEs have to be interpreted correctly at such points according to the specific case. To solve this problem we can replace the conventional derivatives with 'morphological derivatives'. Specifically, we define the morphological *sup-derivative* M^+ a f at a point x as follows:

$$M^+ f(x) \triangleq \lim_{r \to 0} \frac{\sup\{f(x+v) : |v| \leq r\} - f(x)}{r}$$

Similarly, the *inf-derivative* M^- of f is defined as $M^- f(x) = M^+(-f)(x)$ using erosion. Note that $[M^+(f) + M^-(f)]/2$ is equal to Beucher's morphological gradient [8]. It is simple to establish the following:

THEOREM 2 . *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous and let its right* D^+ *and left derivative* D^- *exist at some point* x*. Then (a):*

$$M^+(f)(x) \quad = \quad \begin{cases} \max(|D^-|,|D^+|) & \text{if} \quad D^- \leq D^+ \\ \min(|D^-|,|D^+|) & \text{if} \quad D^- \geq D^+ \end{cases}$$

$$M^-(f)(x) \quad = \quad \begin{cases} \min(|D^-|,|D^+|) & \text{if} \quad D^- \leq D^+ \\ \max(|D^-|,|D^+|) & \text{if} \quad D^- \geq D^+ \end{cases}$$

(b) If $D^+ = D^- = df(x)/dx$*, then* $M^+(f)(x) = M^-(f)(x) = |df(x)/dx|$.

Hence, a more general form of the dilation and erosion PDEs results from replacing $|\partial\alpha/\partial x|$ and $|\partial\beta/\partial x|$ with $M_x^+(\alpha)$ and $M_x^-(\beta)$ respectively, where M_x^+ and M_x^- are the partial morphological sup- and inf-derivatives resulting from applying the definitions of M^+ and M^- in the x direction. These general forms allow the dilation and erosion PDEs to still hold even if discontinuities are created in the partial derivatives $\partial\alpha/\partial x$ and $\partial\beta/\partial x$ as the scale increases, provided that the equations evolve in such way as to give solutions that are piecewise differentiable with left and right limits at each point.

2.2. Dilations/Erosions by 2-D Sets

Given a differentiable function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$, we can find similar PDEs (as in the 1-D case) for its multiscale dilations $\alpha(x,y,s) = f \oplus sB(x,y)$ and erosions $\beta(x,y,s) = f \ominus sB(x,y)$ by a 2-D convex compact structuring set B. The only difference now is that the shape of B affects the form of the PDE. We present results for three different shapes of B: (i) unit *diamond* $\{(u,v) : |v| + |u| \leq 1\}$, (ii) unit *disk* $\{(v,u) : v^2 + u^2 \leq 1\}$, and (iii) unit *square* $\{(v,u) : |v|, |u| \leq 1\}$.

THEOREM 3 . *If the partial derivatives along the x and y directions of α and β exist and are continuous at some point (x,y) and scale s, then at the specific (x,y,s)*

$$\frac{\partial\alpha}{\partial s} = \max\left\{\left|\frac{\partial\alpha}{\partial x}\right|, \left|\frac{\partial\alpha}{\partial y}\right|\right\} \quad , \quad B = \text{diamond}$$

$$\frac{\partial\alpha}{\partial s} = \sqrt{\left|\frac{\partial\alpha}{\partial x}\right|^2 + \left|\frac{\partial\alpha}{\partial y}\right|^2} \quad , \quad B = \text{disk}$$

$$\frac{\partial\alpha}{\partial s} = \left|\frac{\partial\alpha}{\partial x}\right| + \left|\frac{\partial\alpha}{\partial y}\right| \quad , \quad B = \text{square}$$

The PDEs for the multiscale erosions β result from to the above dilation equations by multiplying the right sides with -1.

Proof: Since $\alpha(x+v, y+u, s) - \alpha(x,y,s) = \alpha_x v + \alpha_y u + ||(v,u)|| o(||(u,v)||)$, by ignoring in the limit $r \downarrow 0$ the term with $o()$, it follows that

$$\frac{\partial\alpha}{\partial s} = \lim_{r\downarrow 0} \frac{K}{r} \quad ; \quad K = \sup\{\alpha_x v + \alpha_y u : (v,u) \in rB\}$$

Due to the x,y symmetry of rB, we can replace α_x, α_y with their absolute values and search only over $v \in [0,r]$. Also, due to its linearity, the function $\alpha_x v + \alpha_y u$ over the compact domain rB assumes its maximum value on the boundary of rB. The boundary function $b : [0,r] \rightarrow [0,r]$ is equal to (i) $b(v) = r - v$ if B=diamond, (ii) $b(v) = \sqrt{r^2 - v^2}$ if B=disk, and (iii) $b(v) = r$ if B=square. Hence,

$$\begin{aligned} K &= \max\{|\alpha_x| v + |\alpha_y| b(v) : 0 \leq v \leq r\} \\ &= \begin{cases} r \cdot \max(|\alpha_x|, |\alpha_y|) & , \text{ if } B\text{=diamond} \\ r\sqrt{\alpha_x^2 + \alpha_y^2} & , \text{ if } B\text{=disk} \\ r\,(|\alpha_x| + |\alpha_y|) & , \text{ if } B\text{=square} \end{cases} \end{aligned}$$

This completes the proof of the dilation PDEs. Similarly for β by replacing sup with inf. □

Thus, if $\partial\alpha/\partial x, \partial\alpha/\partial y$ and $\partial\beta/\partial x, \partial\beta/\partial y$ remain continuous for all scales s, we can use the previous PDEs to generate the multiscale ensembles α and β, starting from the initial condition $\alpha(x,y,0) = \beta(x,y,0) = f(x,y)$. Otherwise, if the one-sided x,y partial derivatives of α and β exist everywhere, then we can use the generalized forms of these PDEs where the standard derivatives are replaced by morphological sup- and inf-derivatives, as in the 1-D case.

2.3. Dilations/Erosions by Functions

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable function and let $g : [-1,1] \rightarrow \mathbf{R}$ be a structuring function. Then it is possible to find PDEs for the multiscale dilations $\alpha(x,s) = f \oplus g_s(x)$ and erosions $\beta(x,s) = f \ominus g_s(x)$, which are more general than those in the case $g = 0$. We discuss three different shapes of g: (i) *triangular* $g(x) = g(0)(1 - |x|)$, (ii) *circular* $g(x) = g(0)\sqrt{1 - x^2}$, and (iii) *rectangular* $g(x) = g(0)$, with $g(0) \geq 0$ and $|x| \leq 1$. See Fig. 2.

THEOREM 4 . *If $\partial\alpha/\partial x$ and $\partial\beta/\partial x$ exist at some point x and scale s, then*

$$\frac{\partial\alpha}{\partial s}(x,s) = \max\left\{\left|\frac{\partial\alpha}{\partial x}\right|, g(0)\right\} \quad , \quad g = \text{triangular}$$

$$\frac{\partial\alpha}{\partial s}(x,s) = \sqrt{\left|\frac{\partial\alpha}{\partial x}\right|^2 + g^2(0)} \quad , \quad g = \text{circular}$$

$$\frac{\partial\alpha}{\partial s}(x,s) = \left|\frac{\partial\alpha}{\partial x}\right| + g(0) \quad , \quad g = \text{rectangular}$$

The PDEs for the multiscale erosions β result from to the above dilation equations by multiplying the right sides with -1.

Proof: The proof is very similar to the proof of Theorem 3, because the graph of g is the (scaled by $g(0)$) top boundary of one of the three planar sets used as structuring sets in Theorem 3. □

Note that the PDE for dilation by a set results as a special case of any of the previous three PDEs for dilations by functions g by setting $g(0) = 0$. The PDEs for 1-D dilations and erosions by functions can be easily extended to the 2-D case.

3. OPENING AND CLOSING

The multiscale morphological opening ψ and closing ϕ of f by g are defined as

$$\psi(x,s) \triangleq f \circ g_s(x) = (f \ominus g_s) \oplus g_s(x)$$

$$\phi(x,s) \triangleq f \bullet g_s(x) = (f \oplus g_s) \ominus g_s(x)$$

The simplest way to generate the above multiscale openings and closings using PDEs evolving in scale-signal space would be to implement them serially as compositions of multiscale erosions and dilations. Specifically, $\psi(x,s)$ could be obtained by running the erosion PDE for $\beta(x,r)$ over scales $r \in [0,s]$ with initial condition $\beta(x,0) = f(x)$ and then running the dilation PDE for $\alpha(x,r)$ over scales $r \in [0,s]$ with initial condition $\alpha(x,0) = \beta(x,s)$.

Alternatively, for 1-D openings by sets we have derived the following PDE that directly models the scale evolution of the opening. Consider a differentiable function $f : \mathbf{R} \rightarrow$

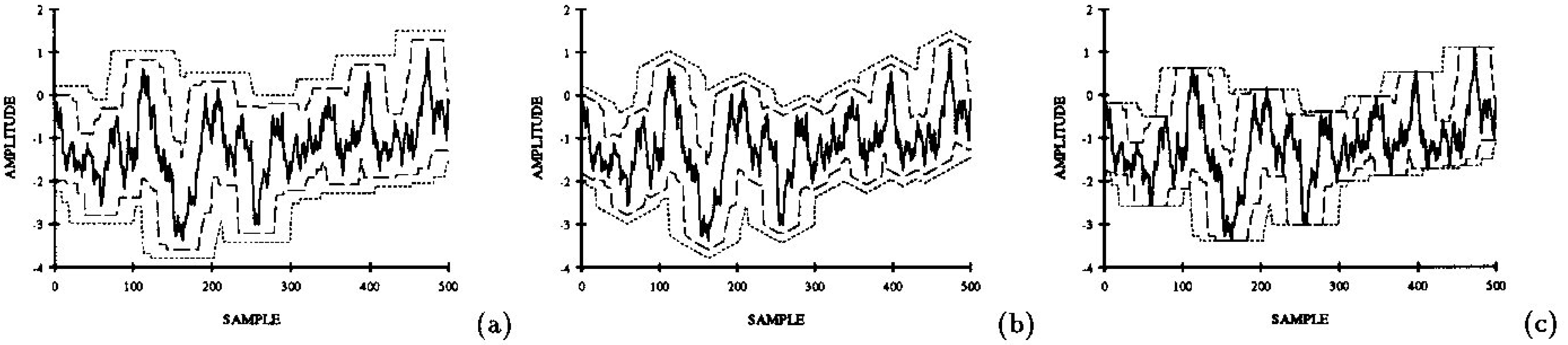

Figure 2: A discrete signal (solid line) and its erosions/dilations (dashed lines) by g_s at scales $s = 20, 40$, where g is a 3-sample sequence. (a) Rectangular g with $g(0) = 0.01$. (b) Triangular g with $g(0) = 0.01$. (c) Rectangular g with $g(0) = 0$.

$\mathbb{R}$ and let $g : B \to \{0\}$ be constant with $B = [-1, 1]$. As s increases the opening $\psi(x, s) = f \circ sB$ becomes smaller, because the peaks of f that have width $\leq 2s$ are cut down. In the process, flat plateaus of length $2s$ are created under the peaks of f. Consider some peak point $x = p$ where f has a local maximum, surrounded by two valley points v_1, v_2 where f has local minima. Let $e_1 \in [v_1, p]$ and $e_2 = e_1 + 2s \in [p, v_2]$ be the left and right end points of x-intervals of length $2s$ such that $\psi(e_1, s) = \psi(e_2, s) = f(e_1)$. These intervals are the supports of the flat plateaus created by the opening. Now let $s \to s + \Delta s$ and $\psi \to \psi + \Delta\psi$. Then $-2\Delta s = \Delta\psi(|\psi(e_1, s)/\partial x|^{-1} + |\psi(e_2, s)/\partial x|^{-1})$. Hence, by letting $\partial\psi/\partial s = \lim_{\Delta s \to 0} \Delta\psi/\Delta s$, we obtain

$$\frac{\partial\psi}{\partial s} = \begin{cases} -2\left(\left|\frac{\partial\psi}{\partial x}(e_1, s)\right|^{-1} + \left|\frac{\partial\psi}{\partial x}(e_2, s)\right|^{-1}\right)^{-1}, & x \in [e_1, e_2] \\ 0, & x \in (v_1, e_1) \cup (e_2, v_2) \end{cases}$$

where $\partial\psi/\partial x$ are one-sided derivatives.

Similarly, by replacing in the opening PDE the -2 with $+2$ and the peak with valley points, a PDE results for the multiscale closing ϕ. Since the opening PDE acts only on the signal's peaks whereas the closing PDE acts only on the valleys, we can also combine both rules into a single PDE that models the evolution of the multiscale *opening-closing*, the composition of opening and closing which smooths signals similarly to a median.

Extending the above opening PDE to 2-D signals f and 2-D sets B presents several problems because the geometry of the 2-D flat plateaus created by the opening are not related to the geometry of the 2-D set B as simply as in the 1-D case.

4. DISCUSSION

The analysis presented here suggests a rather natural way to think about and classify continuous-scale signal operators. If $\mathcal{T}_s(f)$ denotes the output of a multiscale operator at scale s applied to a signal f, then $\mathcal{T}_s$ is said to satisfy the semigroup property if $\mathcal{T}_s[\mathcal{T}_r(f)] = \mathcal{T}_{r+s}(f)$. All the multiscale erosions and dilations discussed in this paper and the convolutions with Gaussians satisfy this property. Multiscale openings and closings, although they do not have an additive semigroup structure, they can be expressed as compositions of operators that do satisfy this rule. Consider next the generator of the semigroup

$$\mathcal{G}[\mathcal{T}_s(f)] = \lim_{r \to 0} \frac{\mathcal{T}_{s+r}(f) - \mathcal{T}_s(f)}{r}$$

If this limit exists in some suitable sense, it may happen that the limit is a differential operator, linear as in the case of Gaussian convolutions, or nonlinear as in the case of dilations/erosions. Alternatively, it may happen that it is a combination of differential and difference operators as we have seen in the case of opening and closing. If it is a differential operator of second order, it will smooth f via diffusion [10]. If it is of first order, the differential equation is of hyperbolic type and $\mathcal{T}_s(f)$ can be expected to evolve by shifting without smoothing.

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