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Multivariate Tropical Regression and Piecewise-Linear Surface Fitting

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Optimal Regression for Fitting Euclidean vs Tropical Lines

Problem: Fit a curve to data $(x_i, y_i), \quad i = 1, \ldots, m$

Euclidean:

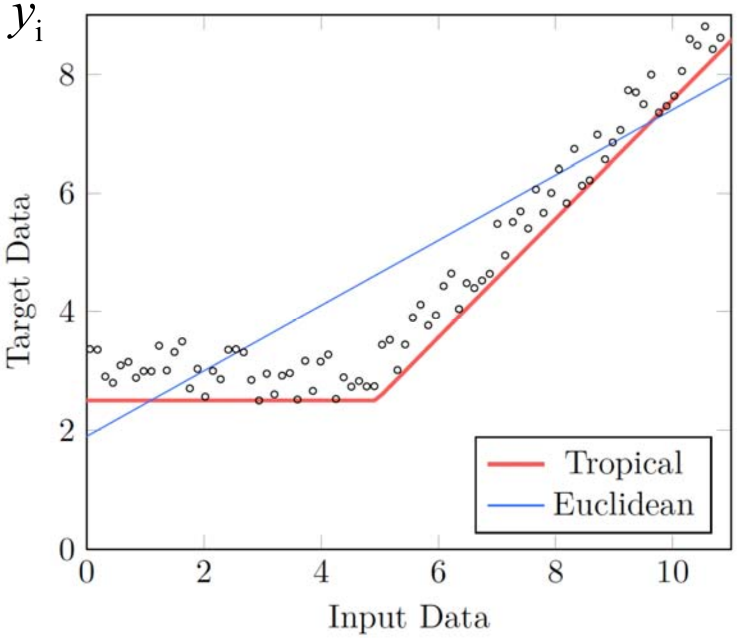
Fit a straight line $y = ax + b$ by minimizing ℓ_2-norm of error:

$$a = \frac{\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)/m}{\sum (x_i)^2 - \left(\sum x_i\right)^2 / m}, \quad b = \frac{1}{m}\sum_i y_i - a x_i$$

Tropical:

Fit a tropical line $y = \max(a + x, b)$ by minimizing some ℓ_p-norm of error:

Greatest Subsolution: $a = \mathrm{MIN}_i\, y_i - x_i \;, \; b = \mathrm{MIN}_i\, y_i$



Outline

1. Elements of Tropical Geometry

- **“a marriage between algebraic geometry and polyhedral geometry”** [Maclagan & Sturmfels 2015]
- **Tropical semirings: Max-plus & Min-plus Arithmetic**
- **Tropical Polynomials**
- **Geometrical objects: tropical lines, tropical polynomial curves/surfaces, tropical half-spaces & polyhedra**

2. Elements of Max-plus Algebra for Vectors/Matrices:

- **Max-plus Matrix algebra**
- **Max-plus systems (Nonlinear Control) and**
- **Signal Processing: Max-plus convolutions (weighted dilations/erosions)**

3. Optimization and Tropical Regression:

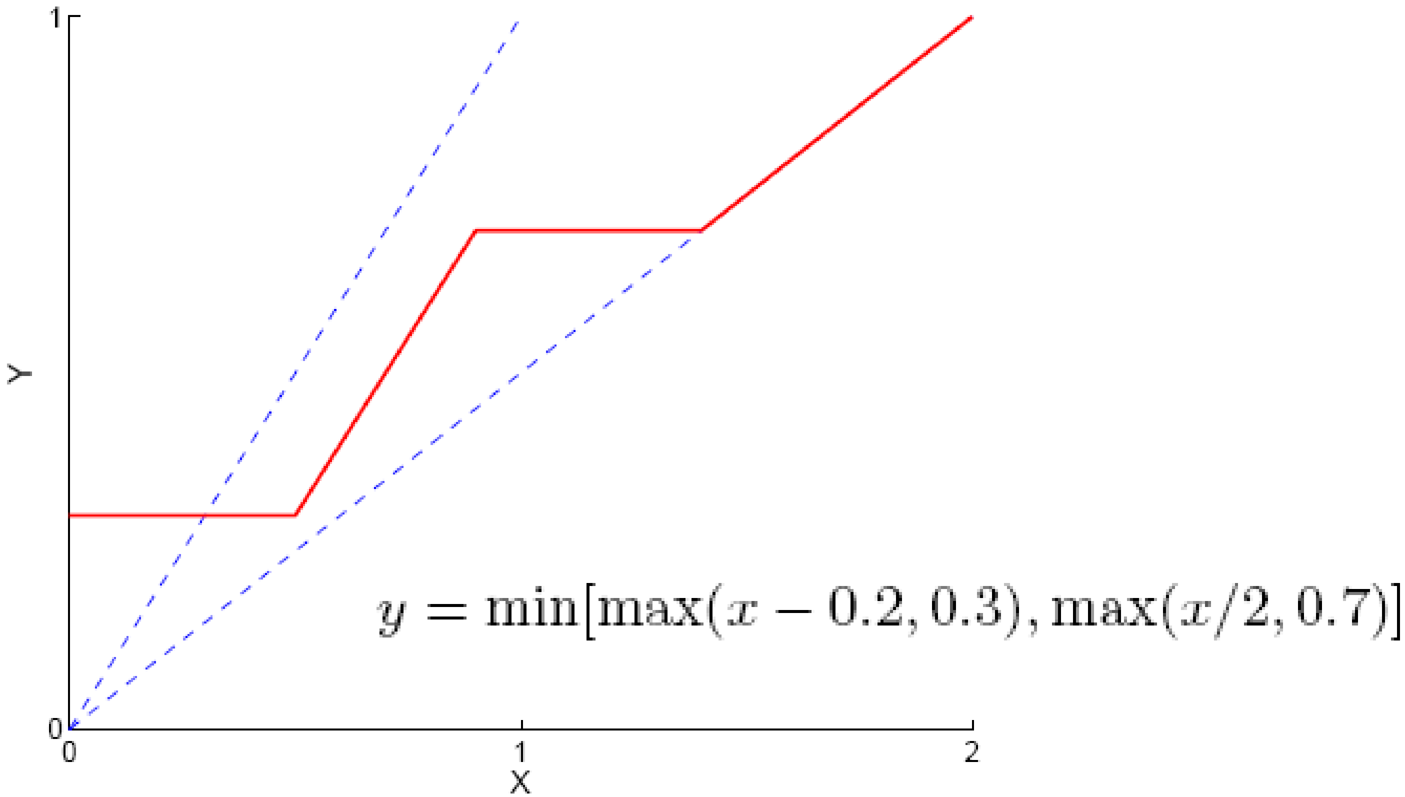
- **Optimal solutions of max-plus matrix equations**
- **Tropical Regression: fitting tropical polynomials to data**
- **Algorithm and Complexity**

What does TROPICAL mean?

- The adjective "tropical" was coined by French mathematicians Dominique Perrin and Jean-Eric Pin, to honor their Brazilian colleague Imre Simon, a pioneer of min-plus algebra as applied to finite automata in computer science.

- Tropical (**Τροπικός** in Greek) comes from the greek word «**Τροπή**» which means "turning" or "changing the way/direction".

Polygonal lines



Tropical Semirings

Scalar Arithmetic Rings

Integer/Real Addition-Multiplication Ring: $(\mathbb{R},+,\times)$, $(\mathbb{Z},+,\times)$

Tropical Semirings

$$\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}, \quad \mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$$

$$\vee = \max, \quad \wedge = \min$$

Max-plus semiring: $(\mathbb{R}_{\max}, \vee, +)$

Min-plus semiring: $(\mathbb{R}_{\min}, \wedge, +)$

Log-Sum-Exp (LSE) approximation

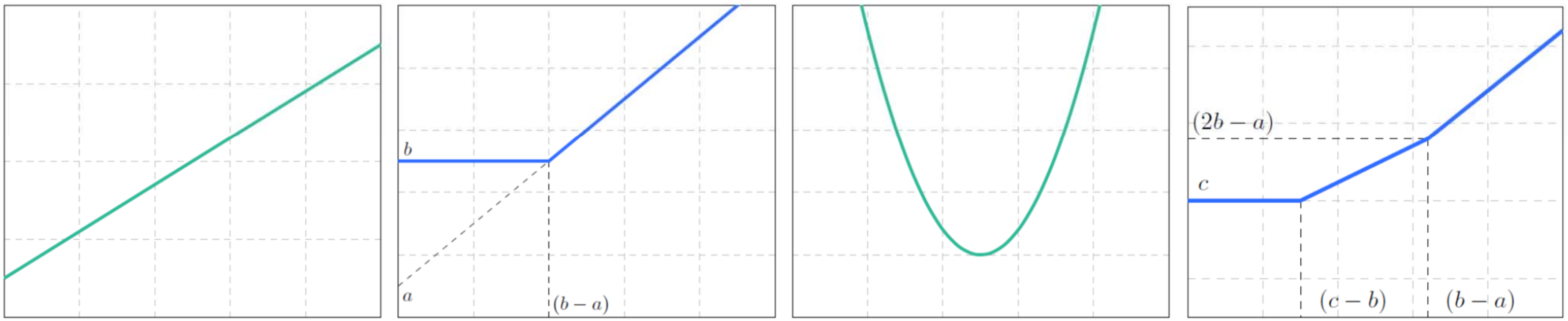
(Maslov "Dequantization" in idempotent mathematics [Maslov 1987, Litvinov 2007])

$$\lim_{T \downarrow 0} T \cdot \log(e^{a/T} + e^{b/T}) = \max(a,b)$$

$$\lim_{T \downarrow 0} (-T) \log(e^{-a/T} + e^{-b/T}) = \min(a,b)$$

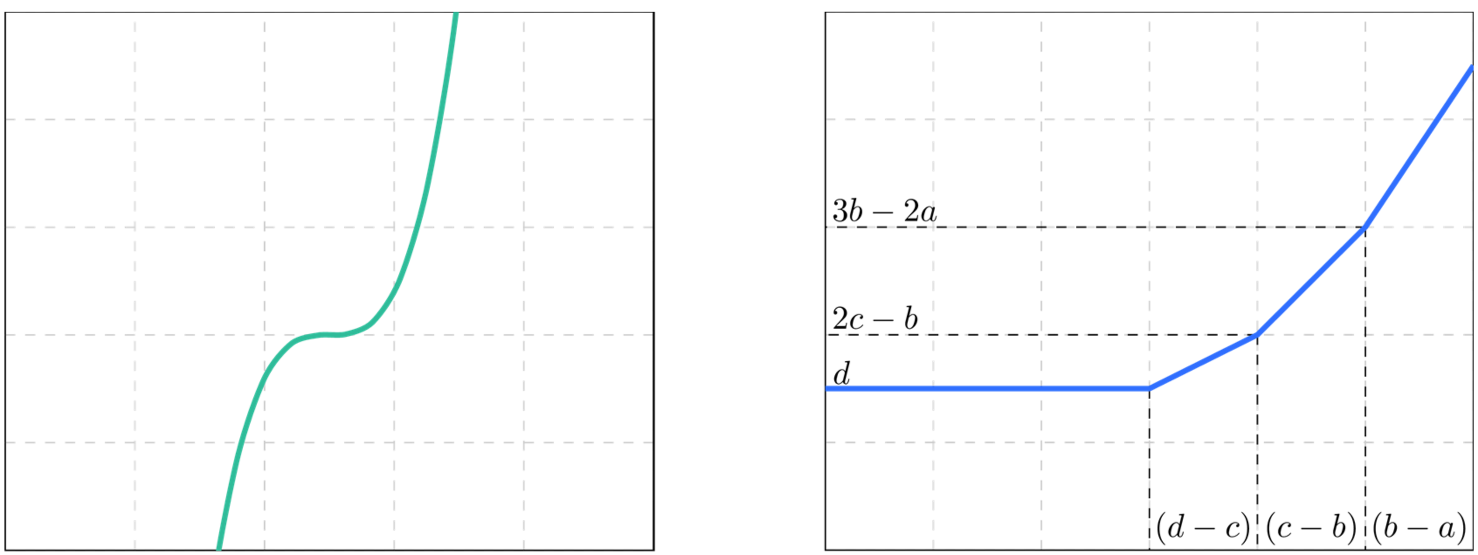
Graphs of Max-plus Tropical 1D Polynomials

$$y_{\text{t-line}} = \max(a + x, b), \quad y_{\text{t-parab}} = \max(a + 2x, b + x, c)$$



(a) Euclidean line (b) Tropical line (c) Euclid parabola (d) Tropic parabola

Cubic polynomial



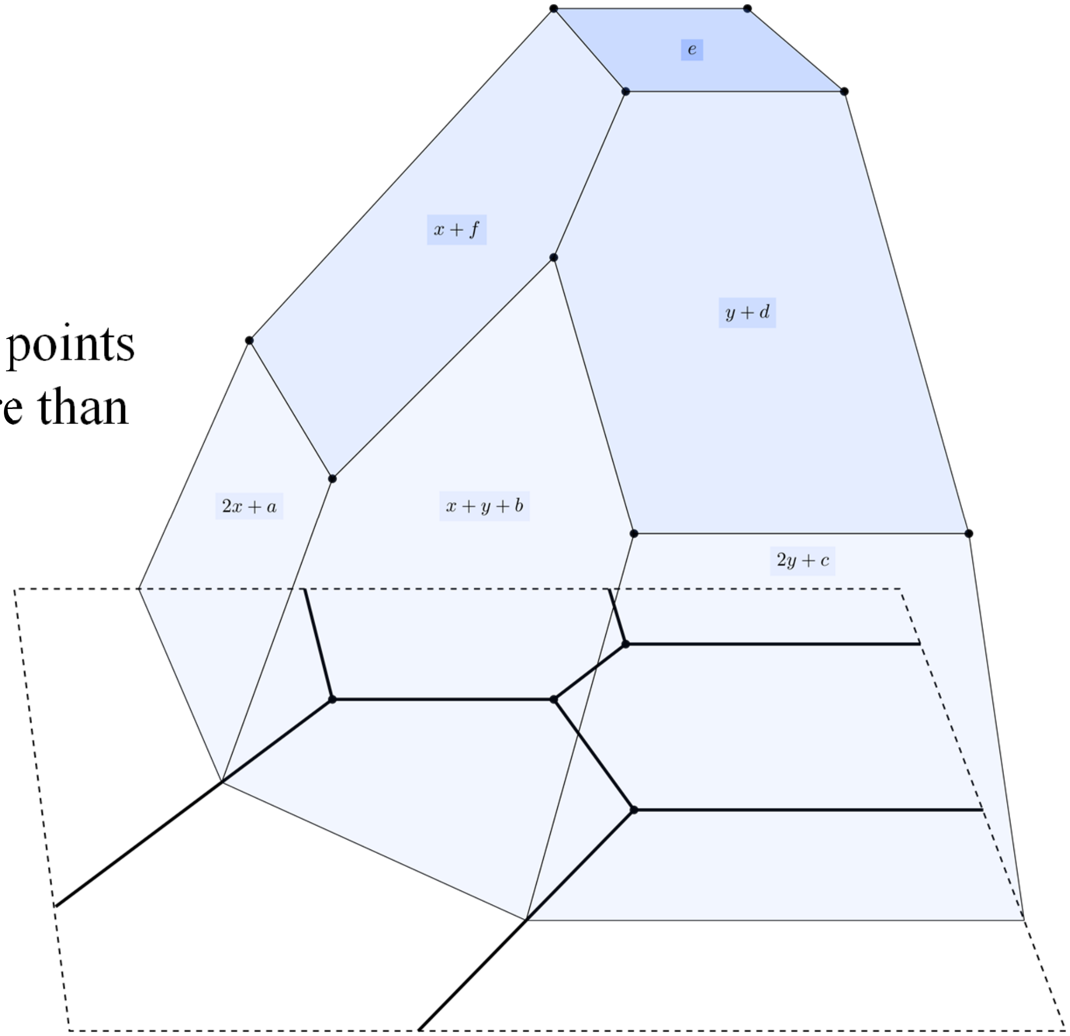
Graph and Trop Curve of a tropical "Conic" polynomial

Tropical Polynomial of degree 2 in two variables

classical: "$ax^2 + bxy + cy^2 + dy + e + fx$"

tropical: $p(x, y) = \min(a + 2x, b + x + y, c + 2y, d + y, e, f + x)$

Graph of $p(x,y)$
and
its **Tropical Curve** = set of (x,y) points where the min is attained by more than one terms.



Obtain Tropical Polynomials via Dequantization

Classic polynomial: $f(\mathbf{u}) = \sum_{k=1}^{K} c_k u_1^{a_{k1}} u_2^{a_{k2}} \cdots u_n^{a_{kn}}, \quad \mathbf{u} = (u_1, u_2, \ldots, u_n)$

Posynomial if $c_k > 0,\ \mathbf{a}_k = (a_{k1}, \ldots, a_{kn}) \in \mathbb{R}^n,\ \mathbf{u} > 0;$

Log-Sum-Exp (Viro's "logarithmic paper" [Viro 2001]):

$\mathbf{x} = \log(\mathbf{u}), \quad b_k = \log(c_k)$

$$\lim_{T \downarrow 0} T \cdot \log f(e^{\mathbf{x}/T}) = \lim_{T \downarrow 0} T \cdot \sum_{k=1}^{K} \exp(\langle \mathbf{a}_k, \mathbf{x}/T \rangle + b_k / T) \ \rightarrow$$

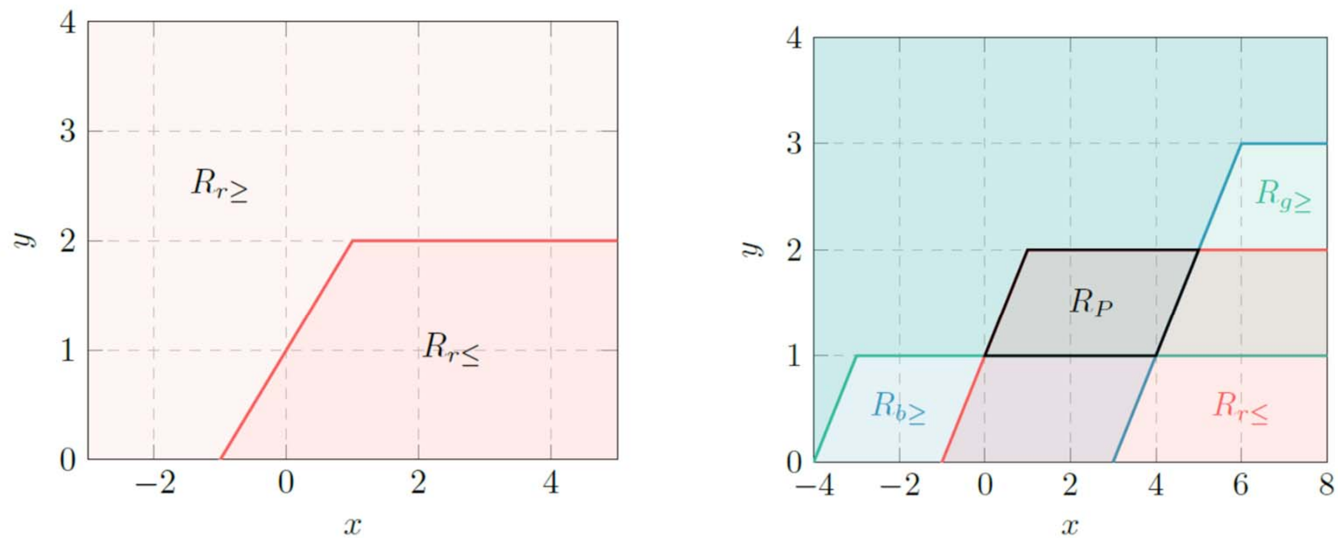
Tropical (max-plus) P**olynomial** = Piecewise-Linear Function

$$p(\mathbf{x}) = \operatorname*{MAX}_{k=1}^{K} \{\langle \mathbf{a}_k, \mathbf{x} \rangle + b_k\} = \operatorname*{MAX}_{k=1}^{K} \{a_{k1} x_1 + \cdots + a_{kn} x_n + b_k\}$$

Tropical Half-spaces and Polytopes in 2D

Tropical (affine) Half-space of $\mathbb{R}^n_{\max}$ [Gaubert & Katz 2011]

$$\mathscr{T}(\mathbf{a},\mathbf{b}) \triangleq \{\mathbf{x} \in \mathbb{R}^n_{\max} : \max(a_{n+1}, \bigvee_{i=1}^{n} a_i + x_i) \leq \max(b_{n+1}, \bigvee_{i=1}^{n} b_i + x_i)\}$$

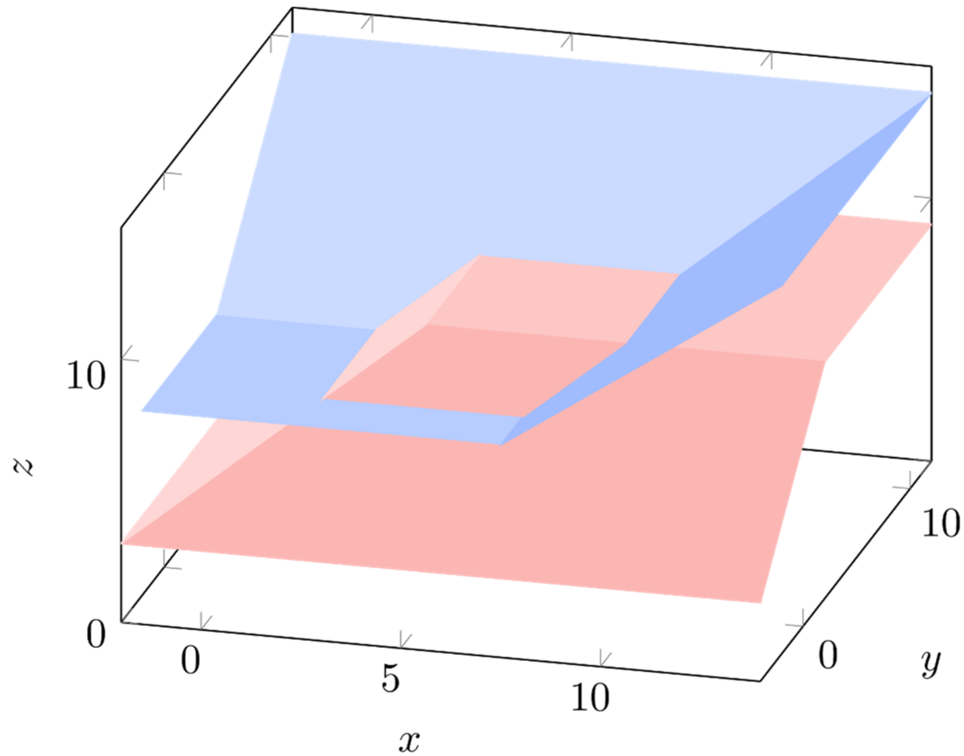


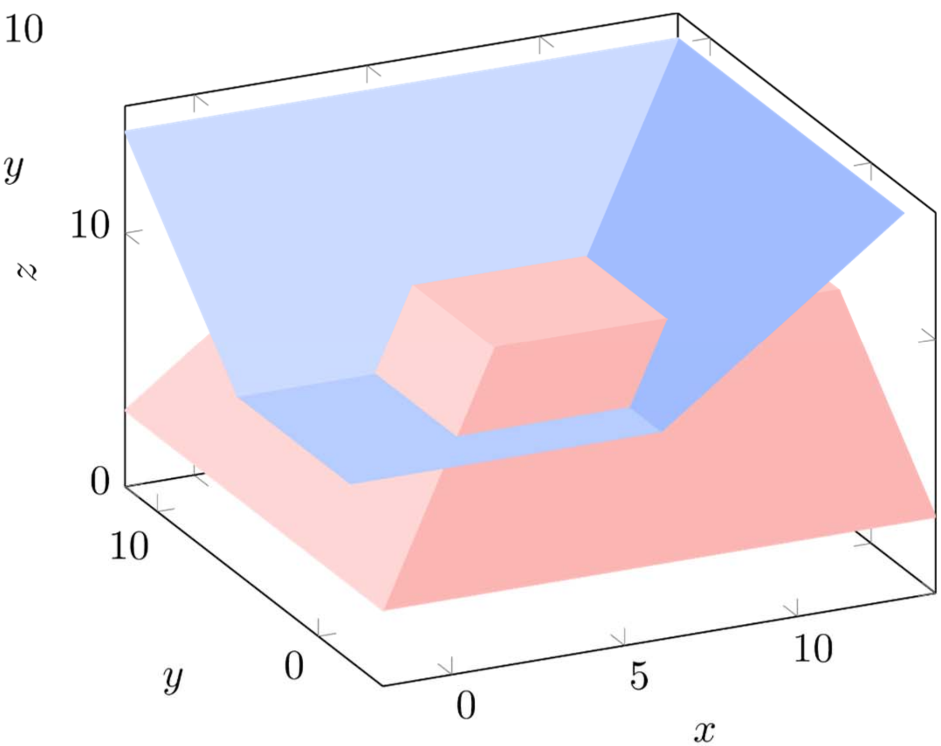
(a) Single region (b) Multiple regions

The region separating boundaries are tropical lines (or hyper-planes).

Tropical **Polyhedra** are formed from finite intersections of tropical half-spaces. **Polytopes** are compact polyhedra.

Tropical Halfspaces and Polyhedra in 3D





Max-plus Matrix Algebra

- vector/matrix **'addition'** = pointwise max

$$\begin{aligned} \mathbf{x} \vee \mathbf{y} &= [x_1 \vee y_1, \ldots, x_n \vee y_n]^T \\ \mathbf{A} \vee \mathbf{B} &= [a_{ij} \vee b_{ij}] \end{aligned}$$

- vector/matrix **'dual addition'** = pointwise min

$$\begin{aligned} \mathbf{x} \wedge \mathbf{y} &= [x_1 \wedge y_1, \ldots, x_n \wedge y_n]^T \\ \mathbf{A} \wedge \mathbf{B} &= [a_{ij} \wedge b_{ij}] \end{aligned}$$

- vector/matrix **'multiplication by scalar'**

$$\begin{aligned} c + \mathbf{x} &= [c + x_1, \ldots, c + x_n]^T \\ c + \mathbf{A} &= [c + a_{ij}] \end{aligned}$$

- $(\max, +)$ **'matrix multiplication'**

$$[\mathbf{A} \boxplus \mathbf{B}]_{ij} = \bigvee_{k=1}^{n} a_{ik} + b_{kj}$$

- $(\min, +)$ **'matrix dual multiplication'**

$$[\mathbf{A} \boxplus' \mathbf{B}]_{ij} = \bigwedge_{k=1}^{n} a_{ik} + b_{kj}$$

Linear versus Max-Plus Systems

- **State-space representation**: linear vs. max-plus

$$x(k) = Ax(k-1) + Bu(k) \qquad x(k) = A \boxplus x(k-1) \vee B \boxplus u(k)$$

$$y(k) = Cx(k) + Du(k) \qquad y(k) = C \boxplus x(k) \vee D \boxplus u(k)$$

- **Matrix products**
  - Linear: $[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$
  - Max-plus: $[A \boxplus B]_{ij} = \bigvee_{k=1}^{n} a_{ik} + b_{kj}$
- **Example**

$$\begin{bmatrix} 4 & -1 \\ 2 & -\infty \end{bmatrix} \boxplus \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \left\{ \begin{array}{r} \max(x+4, y-1) = 3 \\ x+2 = 1 \end{array} \right\} \Longrightarrow \begin{array}{c} x = -1 \\ y \leq 4 \end{array}$$

- What can we model with max-plus systems?

Discrete Event Systems/Control, Scheduling, Shortest Paths on Graphs, Dynamic Programming, WFSTs for Speech recognition, Morphological Filters for Image Processing

Ref: [P. Maragos, "*Dynamical Systems on Weighted Lattices: General Theory*", Math. Control, Signals and Systems, 2017.]

Morphological Operators on Lattices

($\leq$ = partial ordering, V = supremum, Λ = infimum)

- ψ is **increasing** iff $f \leq g \Rightarrow \psi(f) \leq \psi(g)$.
- δ is **dilation** iff $\delta(\vee_i f_i) = \vee_i \delta(f_i)$.
- ε is **erosion** iff $\varepsilon(\wedge_i f_i) = \wedge_i \varepsilon(f_i)$.
- α is **opening** iff increasing and antiextensive $(\alpha(f) \leq f)$,

  and idempotent $(\alpha = \alpha^2)$.
- β is **closing** iff increasing and extensive $(\beta(f) \geq f)$,

  and idempotent $(\beta = \beta^2)$.
- (ε, δ) is **adjunction** iff $\boxed{\delta(g) \leq f \Leftrightarrow g \leq \varepsilon(f).}$

  Then: ε is erosion, δ is dilation,

  $\delta\varepsilon$ is opening (projection), $\varepsilon\delta$ is closing (projection).

(Galois connection)
Residuation pair
("Tropical Adjoints")

[Serra 1988; Heijmans & Ronse 1990]

Solve Max-plus Equations

- **Problems**:
  (1) Exact problem: Solve $\delta_A(\mathbf{x}) = \mathbf{A} \boxplus \mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \overline{\mathbb{R}}^{m \times n}, \ \mathbf{b} \in \overline{\mathbb{R}}^m$

  (2) Approximate Constrained: Min $\|\mathbf{A} \boxplus \mathbf{x} - \mathbf{b}\|_{p=1\dots\infty}$ s.t. $\mathbf{A} \boxplus \mathbf{x} \leq \mathbf{b}$

- **Theorem**: (a) The **greatest (sub)solution** of (1) and unique solution of (2) is

$$\hat{\mathbf{x}} = \varepsilon_A(\mathbf{b}) = \mathbf{A}^* \boxplus' \mathbf{b} = [\bigwedge_i b_i - a_{ij}], \quad \mathbf{A}^* \triangleq -\mathbf{A}^T$$

  and yields the **Greatest Lower Estimate (GLE)** of data $\mathbf{b}$:

$$\delta_A(\varepsilon_A(\mathbf{b})) = \mathbf{A} \boxplus (\mathbf{A}^* \boxplus' \mathbf{b}) \leq \mathbf{b}$$

  (b) **Min Max Absolute Error (MMAE) unconstrained unique solution**:

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} + \mu, \quad \mu = \|\mathbf{A} \boxplus \hat{\mathbf{x}} - \mathbf{b}\|_\infty / 2$$

- **Geometry**: Operators δ, ε are vector dilation and erosion, and
  the GLE $\mathbf{b} \mapsto \delta\varepsilon(\mathbf{b})$ is an opening (lattice projection).

- **Complexity**: $O(mn)$

Optimally Fitting Tropical Lines to Data

Problem: Fit a tropical line $y = \max(a + x, b)$ to noisy data (x_i, f_i), $i = 1, \dots, m$, where $f_i = y_i + \text{error}$ by minimizing $\ell_{1,\dots,\infty}$ norm of error:

Greatest Subsolution (GLE): $\hat{w} = (\hat{a}, \hat{b})$, $\hat{a} = \mathrm{MIN}_i \, f_i - x_i$, $\hat{b} = \mathrm{MIN}_i \, f_\mathrm{i}$

Min Max Abs. Error (MMAE) Solution: $\tilde{w} = \hat{w} + \mu$, $\mu = \|\text{GLE error}\|_\infty /2$

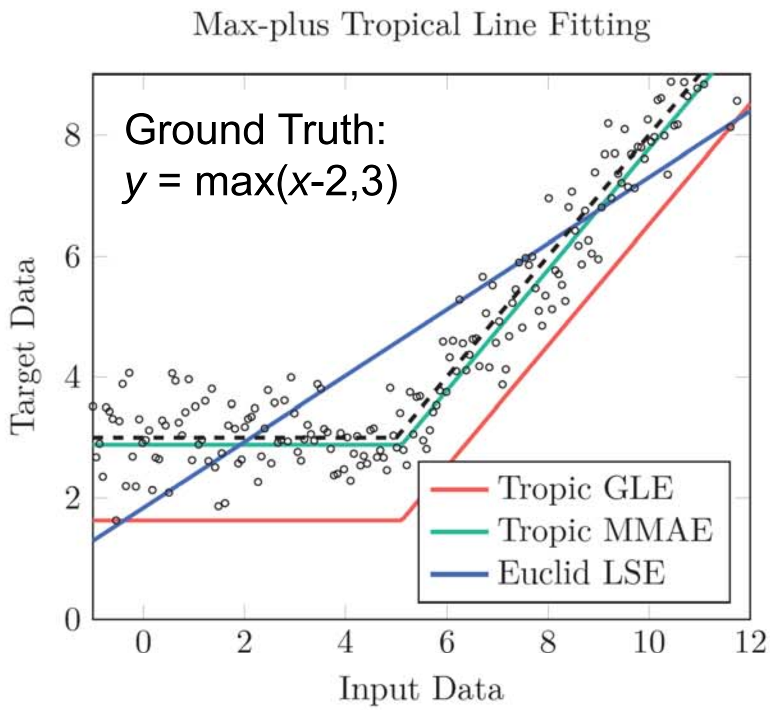
$$\underbrace{\begin{bmatrix} x_1 & 0 \\ \vdots & \vdots \\ x_m & 0 \end{bmatrix}}_{\mathbf{X}} \boxplus \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{\mathbf{f}} \Longrightarrow \underbrace{\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}}_{\hat{\mathbf{w}}} = \underbrace{\begin{bmatrix} \bigwedge_i f_i - x_i \\ \bigwedge_i f_i \end{bmatrix}}_{\mathbf{X}^* \boxplus' \mathbf{f}}$$

Numerical Examples of Optimally Fitting Tropical Lines to Data

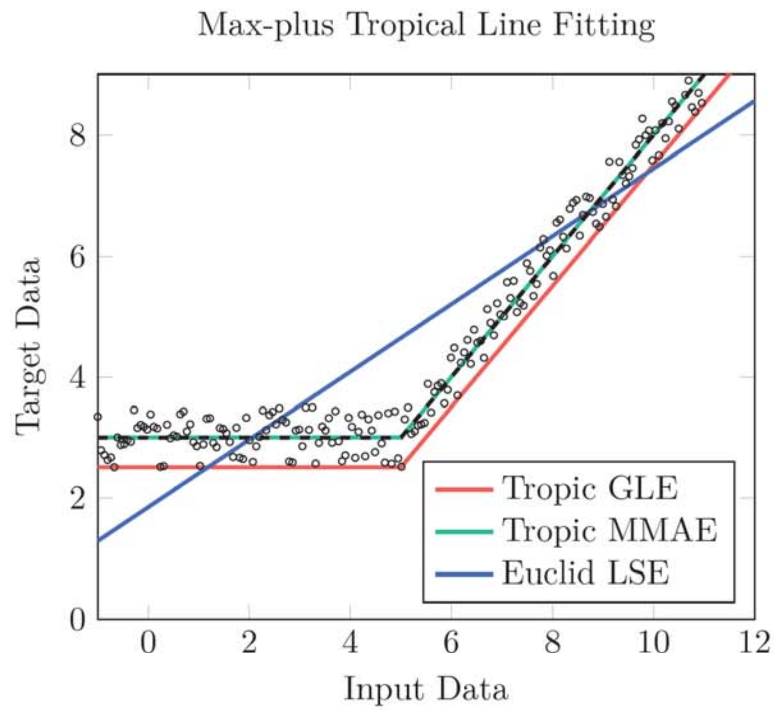
Problem: Fit a tropical line $y = \max(a + x, b)$ to noisy data (x_i, f_i), $i = 1, \ldots, m = 200$, where $f_i = y_i + \text{error}$ by minimizing $\ell_{1,\ldots,\infty}$ of error:

Greatest Subsolution (GLE): $\hat{w} = (\hat{a}, \hat{b})$, $\hat{a} = \mathrm{MIN}_i\, f_i - x_i$, $\hat{b} = \mathrm{MIN}_i\, f_i$

Min Max Abs. Error (MMAE) Solution: $\tilde{w} = \hat{w} + \mu$, $\mu = \|\text{GLE error}\|_\infty /2$



(a) T-line with Gaussian Noise



(b) T-line with Uniform Noise

Optimal Fitting Max-Plus Tropical Planes to Data

Problem: Fit a tropical plane $z = \max(a + x, b + y, c)$ to noisy data (x_i, y_i, f_i), where $f_i = z_i$+error, $i = 1, \ldots, m = 100$, by minimizing $\ell_{1,\ldots,\infty}$ norm of error:

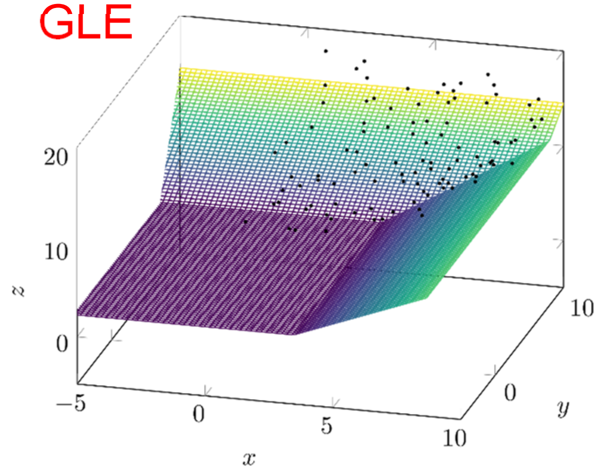
Greatest Subsolution (GLE): $\hat{w} = (\hat{a}, \hat{b}, \hat{c})$

Min Max Abs. Error (MMAE) Solution: $\tilde{w} = \hat{w} + \mu, \quad \mu = \|\text{GLE error}\|_\infty /2$

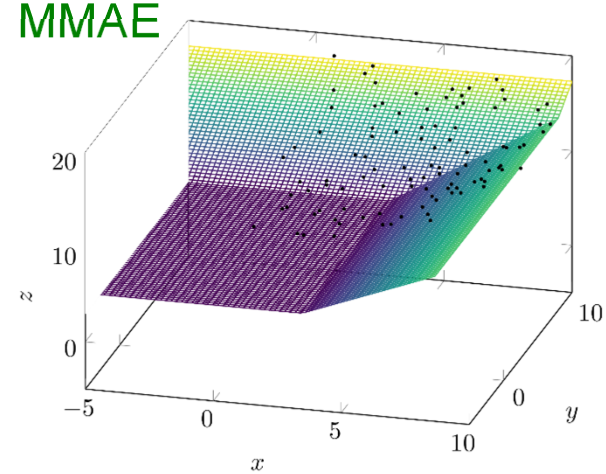
$$\underbrace{\begin{bmatrix} x_1 & y_1 & 0 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 0 \end{bmatrix}}_{\mathbf{X}} \boxplus \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{\mathbf{f}} \Longrightarrow \underbrace{\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}}_{\hat{\mathbf{w}}} = \underbrace{\begin{bmatrix} \bigwedge_i f_i - x_i \\ \bigwedge_i f_i - y_i \\ \bigwedge_i f_i \end{bmatrix}}_{\mathbf{X}^* \boxplus' \mathbf{f}}$$

Ground Truth:
$z = \max(x + 5, y + 7, 9)$
Noise: $N(0,1)$

GLE



MMAE



Optimal Fitting 2D Higher-degree Tropical Polynomials to Data

Data (noisy paraboloid):

3D tuples $(x_i, y_i, f_i) \in \mathbb{R}^3$

$f_i = x_i^2 + y_i^2 + \varepsilon_i,$

$(x_i, y_i) \sim \text{Unif}[-1,1]$

$\varepsilon_i \sim \mathcal{N}(0, 0.25^2)$

Model:

Fit K-rank 2D trop. polynomial

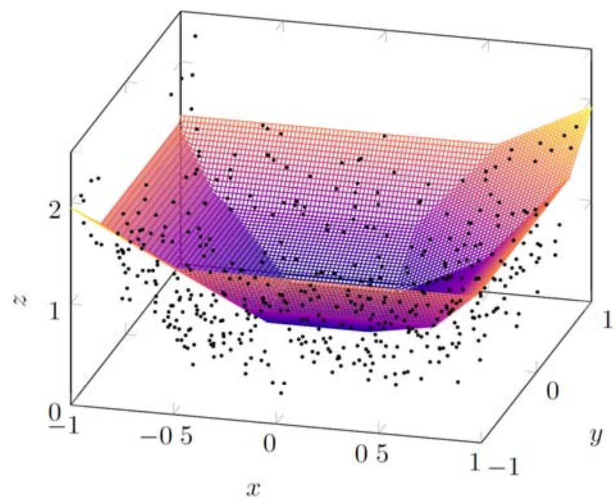
$$p(x, y) = \operatorname*{MAX}_{k=1}^{K} \{ a_k x + b_k y + c_k \}$$

by minimizing error $\| f_i - p(x_i, y_i) \|$。

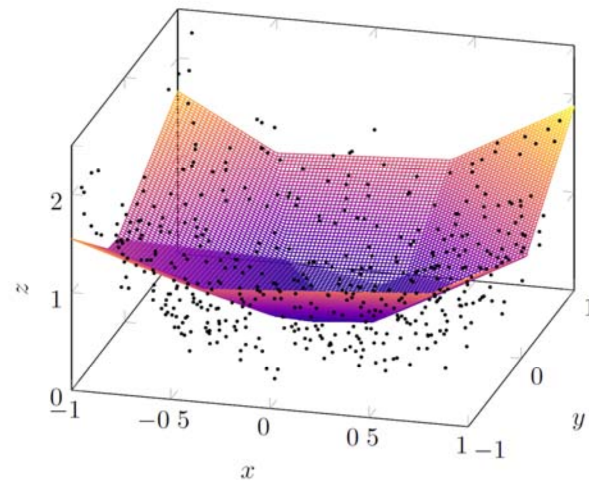
Estimation algorithm:

K – means on data gradients $\rightarrow a_k, b_k$

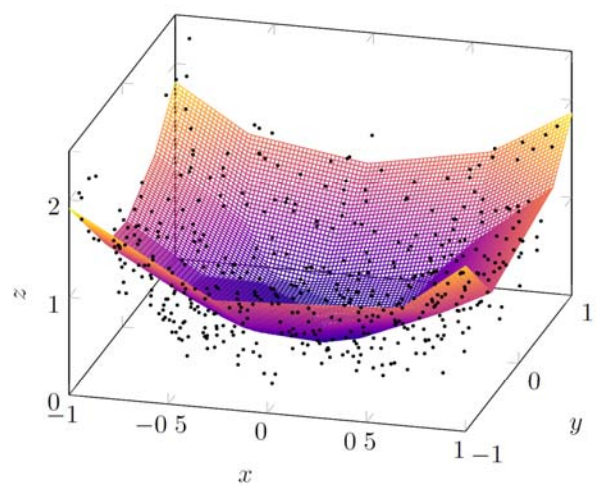
solve max-plus eqns $\rightarrow c_k$



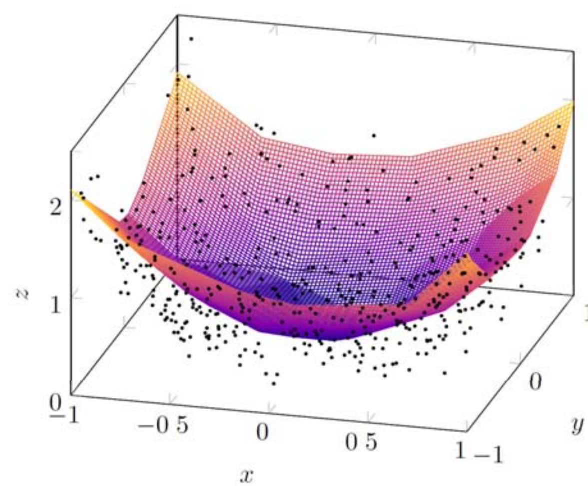
(a) 2D conic (K=11)



(b) K=10



(c) K=25



(d) K=100

Computational Complexity of Tropical Regression

Data size: m data in $\mathbb{R}^{n+1}$

Model: MAX of K hyperplanes $\mathbf{a}_k^T \cdot \mathbf{x} + b_k$, $k = 1, \ldots, K$, $\mathbf{x} \in \mathbb{R}^n$

of **Parameters**: $K(n+1)$

Traditional Least-Squares Estimator: $O((n+1)^3 m^3)$

(Quadratic Programming with Constraints)

Iterative K-means Partition and LSE: $O((n+1)^2 mi)$

[Magnani & Boyd 2009], $i =$ # iterations until convergence

Convex Adaptive Partitioning: $O(n(n+1)^2 m \log(m) \log(\log(m)))$

[Hannah & Dunson 2011]

Our algorithm: $O(Kmni)$

$i =$ # K-means iterations

if $iK \ll m \Rightarrow O(mn)$

Conclusions

- Tropical Geometry, max-plus matrix algebra, and morphological signal operators share a common idempotent semiring arithmetic.

- Introduced Tropical Polynomials for multidimensional data fitting using Piecewise-Linear Functions.

- Developed algorithm of low-complexity (~linear) for tropical regression based on optimal solutions of systems of max-plus equations.

- Future work: extensions to more general regression functions using max-* algebra on weighted lattices.

References

- P. Maragos and E. Theodosis, "*Tropical Geometry and Piecewise-Linear Approximation of Curves and Surfaces on Weighted Lattices*", arXiv, 2019.
- P. Maragos, "*Dynamical Systems on Weighted Lattices: General Theory*", Math. Control, Signals and Systems, 2017.

Thank you for your attention!

We wish everyone courage and health
during the COVID19 pandemic.

For more information, demos, and current results:

http://cvsp.cs.ntua.gr and http://robotics.ntua.gr